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ABSTRACT

We previously eliminated, through Von Zeipel's method, the short period terms of a first order general planetary theory in which we neglect the powers of eccentricities and mutual inclination higher than the third. We enlarge our results by considering n planets instead of two, that is to say $n - 1$ disturbing planets instead of one ($n > 2$), by referring the inclinations to a common fixed plane, the longitudes to a common origin and by introducing the Hori canonical variables instead of the Delaunay canonical variables. We thus eliminate the small divisors which appeared with the Delaunay variables in the partial derivatives of the determining function with respect to the linear variables. On the other hand, the arguments of the sines and cosines of the truncated Fourier series of the disturbing function, of the determining function and of the derivatives of the determining function with respect to the Hori variables are of the form $q\lambda_u - q\lambda_v$, q being a relative integer, which was not the case with the Poincare variables.

ON A GENERALIZATION OF THE ELIMINATION OF THE SHORT PERIOD TERMS OF A FIRST ORDER GENERAL PLANETARY THEORY THROUGH VON ZEIPEL'S METHOD AND HORI CANONICAL VARIABLES

1. INTRODUCTION

The purpose of this paper is to enlarge the results of two previous papers dealing with the elimination, through Von Zeipel's method, of the short period terms of a first order general planetary theory in which we neglect the powers of eccentricities and mutual inclination higher than the third[(*)]. We consider n planets instead of two that is to say $n - 1$ disturbing planets instead of one ($n > 2$); we refer the disturbed planet P_1 to the Sun S, the disturbing planet P_i $(i = 2, 3, \ldots, n)$ to the center of mass of S and P_1, $P_2, \ldots, P_{i-1}$; we refer the inclinations to a common fixed plane, the longitudes to a common origin and we reduce the Fourier series of the principal part F_{1p} of the disturbing function, that of the determining function S_{1p} which corresponds to F_{1p} and that of the partial derivatives of S_{1p} with respect to the canonical variables to the sum of their $p + 1$ first terms, the integer p being unspecified. Moreover, instead of using, as we did in our two previous papers, the Delaunay variables, we use the Hori variables which are defined by the equalites:

$$
\begin{aligned}
&H_u = L_u + M_u + N_u, && \lambda_u, \\
&X_u = \sqrt{-2M_u}\cos(\lambda_u - \varpi_u), && Y_u = \sqrt{-2M_u}\sin(\lambda_u - \varpi_u), \qquad (u = 1, 2, \cdots, n) \\
&P_u = \sqrt{-2N_u}\cos(\lambda_u - \Omega_u), && Q_u = \sqrt{-2N_u}\sin(\lambda_u - \Omega_u)
\end{aligned}
\tag{1}
$$

with $M_u = G_u - L_u$, $N_u = H_u - G_u$, $\lambda_u = \ell_u + g_u + h_u$, L_u, G_u, H_u being the linear Delaunay variables, ℓ_u, g_u, h_u the angular Delaunay variables and ϖ, Ω_u being respectively the longitude of the perihelia and the longitude of the node.

We start from the generalized development of the principal part F_{1p} of the disturbing function that we performed in a recent paper up to the terms of order four with respect to the eccentricities and the sines of inclinations and that we reduce to its terms of order o, 1, 2, 3[ ]. The Newcomb operators $D_{u,v}$ defined in our paper[ ] and which operate on functions of the ratios $\alpha_{u,v}$ of the semi major axis $(u, v = 1, 2, \ldots, n)$ are then replaced by the operators $H_w\ (\partial/\partial H_w)$ $(w = 1, 2, \ldots, n)$ and the truncated Fourier series of F_{1p}, S_{1p} and the partial derivatives of S_{1p} with respect to the canonical variables are performed according to the sines and cosines of the multiples of the pair of angular variables λ_u, λ_v. As in the case of the Poincaré variables, the partial derivatives of S_{1p} with respect to the linear variables do not introduce small divisors as they do in the case of the Delaunay variables. On the other hand, the sum of the multiples of the λ_u and λ_v in the arguments of the sines and cosines of the truncated Fourier series of F_{1p}, S_{1p} and the partial derivatives of S_{1p} with respect to the Hori canonical variables is equal to zero, that is to say that each argument has the form $q\lambda_u - q\lambda_v$, which is not the case if we deal with the Poincaré variables.

2. CANONICITY OF THE HORI VARIABLES

We start from the canonical system of Poincaré variables to which we apply the condition P of Poincaré. In order to shorten the calculation, we consider only one set of Poincaré variables and the corresponding set of the Hori variables. The extension of the proof in the case of several sets of such variables is obvious.

The Poincaré variables are L, x, p, λ, y, q with

$$x = \sqrt{-2M}\cos\bar{\omega}, \quad y = -\sqrt{-2M}\sin\bar{\omega},$$

$$p = \sqrt{-2N}\cos\Omega, \quad q = -\sqrt{-2N}\sin\Omega \qquad (2).$$

The Hori variables are H, X, P, λ, Y, Q obtained from the equality (1) in which we express the index u.

According to the condition P of Poincaré, we have to show that the expression

$$\lambda\, dL + y\, dx + q\, dp - \lambda\, dH - Y\, dX - Q\, dP$$

is an exact differential, that is to say, since $H = L + M + N$, that

$$y\, dx + q\, dp - \lambda\,(dM + dN) - Y\, dX - Q\, dP \qquad (3)$$

is an exact differential. (3) may be written

$$(y\, dx - \lambda\, dM - Y\, dX) + (q\, dp - \lambda\, dN - Q\, dP).$$

We have, according to (1) in which we suppress the index u and according to (2)

$$\begin{aligned} y\, dx - \lambda\, dM - Y\, dX = {} & (-2M\sin^2\bar{\omega} + 2M\sin^2(\lambda - \bar{\omega}))\, d\bar{\omega} \\ & + (\sin\bar{\omega}\cos\bar{\omega} - \lambda + \sin(\lambda - \bar{\omega})\cos(\lambda - \bar{\omega}))\, dM \\ & - 2M\sin^2(\lambda - \bar{\omega})\, d\lambda \end{aligned} \qquad (4)$$

(4) will be an exact differential if and only if:

$$\frac{\partial}{\partial M}(-2M\sin^2\bar{\omega} + 2M\sin^2(\lambda - \bar{\omega})) = \frac{\partial}{\partial\bar{\omega}}(\sin\bar{\omega}\cos\bar{\omega} - \lambda\sin(\lambda - \bar{\omega})\cos(\lambda - \bar{\omega})),$$

$$\frac{\partial}{\partial\lambda}(-2M\sin^2\bar{\omega} + 2M\sin^2(\lambda - \bar{\omega})) = \frac{\partial}{\partial\bar{\omega}}(-2M\sin^2(\lambda - \bar{\omega})),$$

$$\frac{\partial}{\partial\lambda}(\sin\bar{\omega}\cos\bar{\omega} - \lambda + \sin(\lambda - \bar{\omega})\cos(\lambda - \omega)) = \frac{\partial}{\partial M}(-2M\sin^2(\lambda - \bar{\omega})). \qquad (5)$$

A very brief calculation shows that those three conditions (5) are effectively satisfied. (4) is therefore an exact differential.

A similar calculation would show that $q\, dp - \lambda\, dN - Q\, dP$ is also an exact differential. (3) is therefore an exact differential and the Hori variables are canonical.

3. PRELIMINARY CALCULATIONS

Let us consider again the Hori variables defined by the equality (1). From (1) we obtain:

$$L_u = H_u + \frac{X_u^2 + Y_u^2 + P_u^2 + Q_u^2}{2}. \qquad (6)$$

Since we restrict ourselves to a first order theory, we have, according to the notations of our previous paper above mentioned:

$$L_u \sim k\,\beta_u \sqrt{m_0\,a_u}\,,\qquad L_v \sim k\,\beta_v \sqrt{m_0\,a_v} \tag{7}$$

where

$$\alpha_{u,v} = \frac{a_u}{a_v} = \frac{\beta_v^2}{\beta_u^2}\,\frac{L_u^2}{L_v^2}$$

that is to say, according to (6):

$$\alpha_{u,v} = \frac{\beta_v^2}{\beta_u^2}\left(\frac{2H_u + X_u^2 + Y_u^2 + P_u^2 + Q_u^2}{2H_v + X_v^2 + Y_v^2 + P_v^2 + Q_v^2}\right)^2 \tag{8}$$

whence:

$$\frac{\partial\alpha_{u,v}}{\partial H_u} = \frac{4}{2H_u + X_u^2 + Y_u^2 + P_u^2 + Q_u^2}\,\alpha_{u,v} \tag{9}.$$

The Newcomb operator $D_{u,v} = \alpha_{u,v}\,(d/d\alpha_{u,v})$ acting on functions of $\alpha_{u,v}$ and Laplace coefficients which are themselves functions of $\alpha_{u,v}$, we have the equality between operators:

$$\frac{\partial}{\partial H_u} = \frac{\partial\alpha_{u,v}}{\partial H_u}\,\frac{d}{d\alpha_{u,v}} = \frac{\partial\alpha_{u,v}}{\partial H_u}\,\frac{1}{\alpha_{u,v}}\,\alpha_{u,v}\,\frac{d}{d\alpha_{u,v}} = \frac{\partial\alpha_{u,v}}{\partial H_u}\,\frac{1}{\alpha_{u,v}}\,D_{u,v}$$

that is to say, according to (9):

$$D_{u,v} = A\,\frac{\partial}{\partial H_u} \tag{10}$$

with

$$A = \frac{1}{4}\,(2H_u + X_u^2 + Y_u^2 + P_u^2 + Q_u^2) \tag{11}$$

whence:

$$\frac{\partial A}{\partial H_u} = \frac{1}{2}\,,\qquad \frac{\partial^2 A}{\partial H_u^2} = 0,\,\ldots \tag{12}.$$

From (10) and (12) we obtain:

$$D_{u,v}^2 = A^2\,\frac{\partial^2}{\partial H_u^2} + \frac{A}{2}\,\frac{\partial}{\partial H_u} \tag{13},$$

$$D_{u,v}^3 = A^3\,\frac{\partial^3}{\partial H_u^3} + \frac{3A^2}{2}\,\frac{\partial^2}{\partial H_u^2} + \frac{A}{4}\,\frac{\partial}{\partial H_u} \tag{14}.$$

Moreover, from the second equality (7) and from the equality

$$L_v = H_v + \frac{X_v^2 + Y_v^2 + P_v^2 + Q_v^2}{2}$$

we obtain:

$$\frac{1}{a_v} = \frac{4k^2 m_0 \beta_v^2}{(2H_v + X_v^2 + Y_v^2 + P_v^2 + Q_v^2)^2} \tag{15}$$

e_v and γ_v being respectively the eccentricity and the sine of the inclination of the planet P_v, we have:

$$M_v = G_v - L_v = L_v(\sqrt{1-e_v^2} - 1), \quad H_v = G_v\sqrt{1-\gamma_v^2} = L_v\sqrt{1-e_v^2}\sqrt{1-\gamma_v^2}$$

whence:

$$M_v = \frac{H_v(\sqrt{1-e_v^2} - 1)}{\sqrt{1-e_v^2}\sqrt{1-\gamma_v^2}}$$

that is to say, if we neglect the powers of eccentricities and the sines of inclinations higher than the third:

$$M_v = H_v\left(1 + \frac{1}{2}e_v^2 + \frac{1}{2}\gamma_v^2 + \dots\right)\left(-\frac{1}{2}e_v^2 \dots\right) \sim \frac{-1}{2} H_v e_v^2$$

whence:

$$-2M_v \sim H_v e_v^2 .$$

Therefore:

$$X_v \sim e_v\sqrt{H_v}\cos(\lambda_v - \omega_v) \tag{16}$$

We should see, likewise, that

$$N_v \sim \frac{-1}{2} H_v \gamma_v^2$$

whence

$$-2N_v \sim H_v \gamma_v^2 .$$

Therefore:

$$P_v \sim \gamma_v\sqrt{H_v}\cos(\lambda_v - \Omega_v) \tag{17}$$

Similarly:

$$Y_v \sim e_v\sqrt{H_v}\sin(\lambda_v - \omega_v), \quad Q_v \sim \gamma_v\sqrt{H_v}\sin(\lambda_v - \Omega_v) \tag{18}$$

From (16), (17), (18) and the corresponding equalities in the index u, we deduce easily the values of the expressions

$$e_u \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u - \bar\omega_u), \quad \gamma_u \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u - \Omega_u), \quad e_u^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(2\lambda_u - 2\bar\omega_u),$$

$$e_u e_v \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u + \lambda_v - \bar\omega_u - \bar\omega_v), \quad e_u e_v \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u - \lambda_v - \bar\omega_u + \bar\omega_v), \dots, \quad \gamma_u^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(2\lambda_u - 2\Omega_u),$$

$$\gamma_u\gamma_v \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u - \lambda_v - \Omega_u + \Omega_v), \dots, \quad e_u^3 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(3\lambda_u - 3\bar\omega_u), \quad e_u e_v^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(-\lambda_u + 2\lambda_v + \bar\omega_u - 2\bar\omega_v), \dots,$$

$$\gamma_u\gamma_v e_u \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_u - \bar\omega_u + \lambda_u - \Omega_u - \lambda_v + \Omega_v), \quad \gamma_u\gamma_v e_v \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\lambda_v - \bar\omega_v + \lambda_u - \Omega_u - \lambda_v + \Omega_v), \dots$$

On the other hand, (16), (17), (18) and the corresponding equalities in the index u show that $X_v/\sqrt{H_v}$ and $Y_v/\sqrt{H_v}$ have the same order of magnitude as e_v, $X_u/\sqrt{H_u}$ and $Y_u/\sqrt{H_u}$ the same order of magnitude as e_u, $P_v/\sqrt{H_v}$ and $Q_v/\sqrt{H_v}$ the same order of magnitude as γ_v, $P_u/\sqrt{H_u}$ and $Q_u/\sqrt{H_u}$ the same order of magnitude as γ_u. Neglecting the powers of e_u, γ_u, e_v, γ_v and consequently those of X_u, Y_u, P_u, Q_u, X_v, Y_v, P_v, Q_v higher than the third, we have therefore, according to (8) and (15):

$$\alpha_{u,v} \sim \frac{\beta_v^2}{\beta_u^2}\left(\frac{H_u^2}{H_v^2} - \frac{H_u^2}{H_v^3}(X_v^2 + Y_v^2 + P_v^2 + Q_v^2) + \frac{H_u}{H_v^2}(X_u^2 + Y_u^2 + P_u^2 + Q_u^2)\right) \qquad (19),$$

$$\frac{1}{a_v} \sim k^2 m_0 \beta_v^2\left(\frac{1}{H_v^2} - \frac{1}{H_v^3}(X_v^2 + Y_v^2 + P_v^2 + Q_v^2)\right) \qquad (20)$$

and, according to (11):

$$A^2 \sim \frac{1}{4}(H_u^2 + H_u(X_u^2 + Y_u^2 + P_u^2 + Q_u^2)) \qquad (21),$$

$$A^3 \sim \frac{1}{16}(2H_u^3 + 3H_u^2(X_u^2 + Y_u^2 + P_u^2 + Q_u^2)) \qquad (22).$$

4. EXPRESSION OF THE PRINCIPAL PART F_{1p} OF THE DISTURBING FUNCTION

It is obtained from the expression of F_{1p} calculated in our previous paper (pages 4 to 29)( ) truncated to its terms of order 0, 1, 2, 3 in e_u, e_v, γ_u, γ_v, in which

$$e_u \begin{matrix}\sin\\\cos\end{matrix}(\lambda_u - \bar\omega_u),\quad e_v \begin{matrix}\sin\\\cos\end{matrix}(\lambda_v - \bar\omega_v),\quad \gamma_u \begin{matrix}\sin\\\cos\end{matrix}(\lambda_u - \Omega_u),\quad \gamma_v \begin{matrix}\sin\\\cos\end{matrix}(\lambda_v - \Omega_v),\ \ldots,$$

$$\gamma_u \gamma_v e_v \begin{matrix}\sin\\\cos\end{matrix}(\lambda_v - \bar\omega_v + \lambda_u - \Omega_u - \lambda_v + \Omega_v)\ldots$$

are replaced by their values obtained from (16), (17), (18) and in which $D_{u,v}$, $D^2_{u,v}$, $D^3_{u,v}$, A, A^2, A^3, $\alpha_{u,v}$, $1/a_v$ are replaced by their values (10), (13), (14), (11), (21), (22), (19), (20). We have

$$F_{1p} = \sigma k^4 m_0 \sum_{\substack{u\neq v\\ 1\leq u< v\leq n}} \beta_u \beta_v^3 \sum_{j=0}^{p}\left[\left\{\left(\frac{1}{H_v^2} - \frac{1}{H_v^3}(X_v^2 + Y_v^2 + P_v^2 + Q_v^2)\right) b_{1/2}^{(j,u,v)}\right.\right.$$

$$+\left(\frac{X_u^2 + Y_u^2}{H_u H_v^2} + \frac{X_v^2 + Y_v^2}{H_v^3}\right)\left(-j^2 + \frac{3}{16}H_u\frac{\partial}{\partial H_u} + \frac{1}{16}H_u^2\frac{\partial^2}{\partial H_u^2}\right) b_{1/2}^{(j,u,v)}$$

$$\left.+\left(\frac{-P_u^2 - Q_u^2}{8H_u H_v^4} + \frac{-P_v^2 - Q_v^2}{8H_v^5}\right)\frac{\beta_v^2}{\beta_u^2} H_u^2 \left(b_{3/2}^{(j-1,u,v)} + b_{3/2}^{(j+1,u,v)}\right)\right\}\cos(j\lambda_u - j\lambda_v)$$

$$+\left\{\frac{1}{\sqrt{H_u}\,H_v^2}\left(j-\frac{j}{H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right.\right.$$

$$\left.+\left(\frac{-1}{8}(2H_u+X_u^2+Y_u^2 P_u^2+Q_u^2)+\frac{H_u}{4H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right)\frac{\partial}{\partial H_u}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_u^2+Y_u^2}{H_u\sqrt{H_u}\,H_v^2}\left(-\frac{1}{8}j-\frac{5}{8}j^2-\frac{1}{2}j^3+\left(\frac{13}{128}+\frac{3}{16}j+\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}+\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v^2+Y_v^2}{H_u\sqrt{H_u}\,H_v^2}\left(-j^3+\left(\frac{-3}{64}+\frac{3}{16}j+\frac{1}{4}j^2\right)H_u\frac{\partial}{\partial H_u}+\left(\frac{-5}{64}+\frac{1}{16}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}\right.$$

$$\left.-\frac{1}{64}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\left.+\left(\frac{P_u^2+Q_u^2}{4H_u\sqrt{H_u}\,H_v^4}+\frac{P_v^2+Q_v^2}{4H_v^5\sqrt{H_v}}\right)\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{2}j+\frac{1}{8}H_u\frac{\partial}{\partial H_u}\right)H_u^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)\right\}$$

$$\times\left(X_u\cos(j\lambda_u-j\lambda_v)-Y_u\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\left\{\frac{1}{\sqrt{H_u}\,H_v^2}\left(-j+\frac{j}{H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right.\right.$$

$$\left.+\left(\frac{-1}{8}(2H_u+X_u^2+Y_u^2+P_u^2+Q_u^2)+\frac{H_u}{4H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right)\frac{\partial}{\partial H_u}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_u^2+Y_u^2}{H_u\sqrt{H_u}\,H_v^2}\left(\frac{1}{8}j-\frac{5}{8}j^2+\frac{1}{2}j^3+\left(\frac{13}{128}-\frac{3}{16}j+\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v^2+Y_v^2}{H_u\sqrt{H_u}\,H_v^2}\left(j^3+\left(\frac{-3}{64}+\frac{3}{16}j+\frac{1}{4}j^2\right)H_u\frac{\partial}{\partial H_u}+\left(\frac{-5}{64}-\frac{1}{16}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}\right.$$

$$\left.-\frac{1}{64}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\left(\frac{P_u^2+Q_u^2}{4H_u\sqrt{H_u}\,H_v^4}+\frac{P_v^2+Q_v^2}{4H_v^5\sqrt{H_v}}\right)\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{2}j+\frac{1}{8}H_u\frac{\partial}{\partial H_u}\right)H_u^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)\Bigg\}$$

$$\times\left(X_u\cos(j\lambda_u-j\lambda_v)+Y_u\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\Bigg\{\frac{1}{\sqrt{H_v}\,H_v^2}\left(\frac{1}{2}-j-\frac{1}{H_v}\left(\frac{1}{2}-j\right)\right)(X_v^2+Y_v^2+P_v^2+Q_v^2)$$

$$+\left(\frac{1}{8}(2H_u+X_u^2+Y_u^2+P_u^2+Q_u^2)-\frac{H_u}{4H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right)\frac{\partial}{\partial H_u}\Bigg)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_u^2+Y_u^2}{H_u\sqrt{H_v}\,H_v^2}\left(-\frac{1}{2}j^2+j^3+\left(\frac{9}{64}-\frac{3}{16}j-\frac{1}{4}j^2\right)H_u\frac{\partial}{\partial H_u}+\left(\frac{7}{64}-\frac{1}{16}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}\right.$$

$$\left.+\frac{1}{64}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v^2+Y_v^2}{H_v^3\sqrt{H_v}}\left(-\frac{1}{16}+\frac{5}{16}j-\frac{7}{8}j^2+\frac{1}{2}j^3+\left(\frac{17}{128}-\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{11}{128}-\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\left(\frac{P_u^2+Q_u^2}{4H_u\sqrt{H_v}\,H_v^4}+\frac{P_v^2+Q_v^2}{4H_v^5\sqrt{H_v}}\right)\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{4}+\frac{1}{2}j-\frac{1}{8}H_u\frac{\partial}{\partial H_u}\right)H_u^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)\Bigg\}$$

$$\times\left(X_v\cos(j\lambda_u-j\lambda_v)-Y_v\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\Bigg\{\frac{1}{\sqrt{H_v}\,H_v^2}\left(\frac{1}{2}+j-\frac{1}{H_v}\left(\frac{1}{2}+j\right)\right)(X_v^2+Y_v^2+P_v^2+Q_v^2)$$

$$+\left(\frac{1}{8}(2H_u+X_u^2+Y_u^2+P_u^2+Q_u^2)-\frac{H_u}{4H_v}(X_v^2+Y_v^2+P_v^2+Q_v^2)\right)\frac{\partial}{\partial H_u}\Bigg)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_u^2+Y_u^2}{H_u\sqrt{H_v}\,H_v^2}\left(-\frac{1}{2}j^2-j^3+\left(\frac{9}{64}+\frac{3}{16}j-\frac{1}{4}j^2\right)H_u\frac{\partial}{\partial H_u}+\left(\frac{7}{64}+\frac{1}{16}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}\right.$$

$$\left.+\frac{1}{64}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v^2+Y_v^2}{H_v^3\sqrt{H_v}}\left(-\frac{1}{16}-\frac{5}{16}j-\frac{7}{8}j^2-\frac{1}{2}j^3+\left(\frac{17}{128}-\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{11}{128}+\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\left.+\left(\frac{P_u^2+Q_u^2}{4H_u\sqrt{H_v}H_v^4}+\frac{P_v^2+Q_v^2}{4H_v^5\sqrt{H_v}}\right)\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{4}-\frac{1}{2}j-\frac{1}{8}H_u\frac{\partial}{\partial H_u}\right)H_u^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)\right\}$$

$$\times\left(X_v\cos(j\lambda_u-j\lambda_v)+Y_v\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{H_u H_v^2}\left(\frac{5}{8}j+\frac{1}{2}j^2+\left(\frac{-5}{32}-\frac{1}{4}j\right)H_u\frac{\partial}{\partial H_u}+\frac{1}{32}H_u^2\frac{\partial^2}{\partial H_u^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times((X_u^2-Y_u^2)\cos(j\lambda_u-j\lambda_v)-2X_uY_u\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{H_u H_v^2}\left(-\frac{5}{8}j+\frac{1}{2}j^2+\left(\frac{-5}{32}+\frac{1}{4}j\right)H_u\frac{\partial}{\partial H_u}+\frac{1}{32}H_u^2\frac{\partial^2}{\partial H_u^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times((X_u^2-Y_u^2)\cos(j\lambda_u-j\lambda_v)+2X_uY_u\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}H_v^2}\left(\frac{1}{2}j-j^2+\left(-\frac{3}{16}+\frac{1}{2}j\right)H_u\frac{\partial}{\partial H_u}-\frac{1}{16}H_u^2\frac{\partial^2}{\partial H_u^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times((X_uX_v-Y_uY_v)\cos(j\lambda_u-j\lambda_v)-(X_uY_v+Y_uX_v)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}H_v^2}\left(\frac{1}{2}j+j^2-\frac{3}{16}H_u\frac{\partial}{\partial H_u}-\frac{1}{16}H_u^2\frac{\partial^2}{\partial H_u^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times((X_uX_v+Y_uY_v)\cos(j\lambda_u-j\lambda_v)+(X_uY_v-Y_uX_v)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}H_v^2}\left(-\frac{1}{2}j+j^2-\frac{3}{16}H_u\frac{\partial}{\partial H_u}-\frac{1}{16}H_u^2\frac{\partial^2}{\partial H_u^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times((X_uX_v+Y_uY_v)\cos(j\lambda_u-j\lambda_v)+(-X_uY_v+Y_uX_v)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,\sqrt{H_v}\,H_v^2}\left(-\frac{1}{2}j - j^2 + \left(-\frac{3}{16}-\frac{1}{2}j\right) H_u \frac{\partial}{\partial H_u} - \frac{1}{16} H_u^2 \frac{\partial^2}{\partial H_u^2}\right) b_{1/2}^{(j,u,v)}$$

$$\times\Big((X_u X_v - Y_u Y_v)\cos(j\lambda_u - j\lambda_v) + (X_u Y_v + Y_u X_v)\sin(j\lambda_u - j\lambda_v)\Big)$$

$$+\frac{1}{H_v^3}\left(\frac{1}{2}-\frac{9}{8}j + \frac{1}{2}j^2 + \left(\frac{11}{32}-\frac{1}{4}j\right) H_u \frac{\partial}{\partial H_u} + \frac{1}{32} H_u^2 \frac{\partial^2}{\partial H_u^2}\right) b_{1/2}^{(j,u,v)}$$

$$\times\left((X_v^2 - Y_v^2)\cos(j\lambda_u - j\lambda_v) - 2X_v Y_v \sin(j\lambda_u - j\lambda_v)\right)$$

$$+\frac{1}{H_v^3}\left(\frac{1}{2}+\frac{9}{8}j + \frac{1}{2}j^2 + \left(\frac{11}{32}+\frac{1}{4}j\right) H_u \frac{\partial}{\partial H_u} + \frac{1}{32} H_u^2 \frac{\partial^2}{\partial H_u^2}\right) b_{1/2}^{(j,u,v)}$$

$$\times\left((X_v^2 - Y_v^2)\cos(j\lambda_u - j\lambda_v) + 2X_v Y_v \sin(j\lambda_u - j\lambda_v)\right)$$

$$+\frac{1}{H_u\sqrt{H_u}\,H_v^2}\left(\frac{13}{24}j + \frac{5}{8}j^2 + \frac{1}{6}j^3 + \left(\frac{-17}{128}-\frac{5}{16}j - \frac{1}{8}j^2\right) H_u \frac{\partial}{\partial H_u}\right.$$

$$\left. + \left(\frac{5}{128}+\frac{1}{32}j\right) H_u^2 \frac{\partial^2}{\partial H_u^2} - \frac{1}{384} H_u^3 \frac{\partial^3}{\partial H_u^3}\right) b_{1/2}^{(j,u,v)}$$

$$\times\left(X_u (X_u^2 - 3Y_u^2)\cos(j\lambda_u - j\lambda_v) - Y_u (3X_u^2 - Y_u^2)\sin(j\lambda_u - j\lambda_v)\right)$$

$$+\frac{1}{H_u\sqrt{H_u}\,H_v^2}\left(\frac{-13}{24}j + \frac{5}{8}j^2 - \frac{1}{6}j^3 + \left(\frac{-17}{128}+\frac{5}{16}j - \frac{1}{8}j^2\right) H_u \frac{\partial}{\partial H_u}\right.$$

$$+\left(\frac{5}{128}-\frac{1}{32}j\left(H_u^2 \frac{\partial^2}{\partial H_u^2} - \frac{1}{384} H_u^3 \frac{\partial^3}{\partial H_u^3}\right) b_{1/2}^{(j,u,v)}\right.$$

$$\times\left(X_u (X_u^2 - 3Y_u^2)\cos(j\lambda_u - j\lambda_v) + Y_u (3X_u^2 - Y_u^2)\sin(j\lambda_u - j\lambda_v)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^2}\left(\frac{5}{16}j-\frac{3}{8}j^2-\frac{1}{2}j^3+\left(\frac{-15}{128}+\frac{1}{8}j+\frac{3}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{3}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\Big(((X_u^2-Y_u^2)X_v-2X_uY_uY_v)\cos(j\lambda_u-j\lambda_v)-(2X_uY_uX_v+(X_u^2-Y_u^2)Y_v)\sin(j\lambda_u-j\lambda_v)\Big)$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^2}\left(\frac{-5}{16}j-\frac{3}{8}j^2+\frac{1}{2}j^3+\left(\frac{-15}{128}-\frac{1}{8}j+\frac{3}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}+\frac{3}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\Big(((X_u^2-Y_u^2)X_v-2X_uY_uY_v)\cos(j\lambda_u-j\lambda_v)+(2X_uY_uX_v+(X_u^2-Y_u^2)Y_v)\sin(j\lambda_u-j\lambda_v)\Big)$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^2}\left(\frac{5}{16}j+\frac{7}{8}j^2+\frac{1}{2}j^3+\left(\frac{-15}{128}-\frac{3}{16}j-\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\Big(((X_u^2-Y_u^2)X_v+2X_uY_uY_v)\cos(j\lambda_u-j\lambda_v)+(-2X_uY_uX_v+(X_u^2-Y_u^2)Y_v)\sin(j\lambda_u-j\lambda_v)\Big)$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^2}\left(\frac{-5}{16}j+\frac{7}{8}j^2-\frac{1}{2}j^3+\left(\frac{-15}{128}+\frac{3}{16}j-\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-1}{128}+\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\Big(((X_u^2-Y_u^2)X_v+2X_uY_uY_v)\cos(j\lambda_u-j\lambda_v)+(2X_uY_uX_v-(X_u^2-Y_u^2)Y_v)\sin(j\lambda_u-j\lambda_v)\Big)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^3}\left(\frac{1}{2}j-\frac{9}{8}j^2+\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{11}{16}j-\frac{3}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-13}{128}+\frac{3}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u(X_v^2-Y_v^2)-2Y_uX_vY_v)\cos(j\lambda_u-j\lambda_v)-(Y_u(X_v^2-Y_v^2)+2X_uX_vY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^3}\left(\frac{1}{2}j+\frac{9}{8}j^2+\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-13}{128}-\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u(X_v^2-Y_v^2)+2Y_uX_vY_v)\cos(j\lambda_u-j\lambda_v)+(-Y_u(X_v^2-Y_v^2)+2X_uX_vY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^3}\left(\frac{-1}{2}j+\frac{9}{8}j^2-\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{1}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-13}{128}+\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u(X_v^2-Y_v^2)+2Y_uX_vY_v)\cos(j\lambda_u-j\lambda_v)+(Y_u(X_v^2-Y_v^2)-2X_uX_vY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^3}\left(\frac{-1}{2}j-\frac{9}{8}j^2-\frac{1}{2}j^3+\left(\frac{-27}{128}-\frac{11}{16}j-\frac{3}{8}j^2\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{-13}{128}-\frac{3}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}-\frac{1}{128}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u(X_v^2-Y_v^2)-2Y_uX_vY_v)\cos(j\lambda_u-j\lambda_v)+(Y_u(X_v^2-Y_v^2)+2X_uX_vY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{H_v^3\sqrt{H_v}}\left(\frac{27}{48}-\frac{65}{48}j+\frac{7}{8}j^2-\frac{1}{6}j^3+\left(\frac{59}{128}-\frac{1}{2}j\right)H_u\frac{\partial}{\partial H_u}\right.$$

$$\left.+\left(\frac{9}{128}-\frac{1}{32}j\right)H_u^2\frac{\partial^2}{\partial H_u^2}+\frac{1}{384}H_u^3\frac{\partial^3}{\partial H_u^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\left(X_v\left(X_v^2-3Y_v^2\right)\cos\left(j\lambda_u-j\lambda_v\right)-Y_v\left(3X_v^2-Y_v^2\right)\sin\left(j\lambda_u-j\lambda_v\right)\right)$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}H_v^4}\frac{1}{4}\frac{\beta_v^2}{\beta_u^2}H_u^2\,b_{3/2}^{(j-1,u,v)}\left(\left(P_uP_v+Q_uQ_v\right)\cos\left((j-1)\lambda_u-(j-1)\lambda_v\right)\right.$$

$$\left.-\left(Q_uP_v-P_uQ_v\right)\sin\left((j-1)\lambda_u-(j-1)\lambda_v\right)\right)$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}H_v^4}\frac{1}{4}\frac{\beta_v^2}{\beta_u^2}H_u^2\,b_{3/2}^{(j+1,u,v)}\left(\left(P_uP_v+Q_uQ_v\right)\cos\left((j+1)\lambda_u-(j+1)\lambda_v\right)\right.$$

$$\left.+\left(Q_uP_v-P_uQ_v\right)\sin\left((j+1)\lambda_u-(j+1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}H_v^4}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j-1,u,v)}$$

$$\times\left(\left(X_u\left(P_uP_v+Q_uQ_v\right)+Y_u\left(P_uQ_v-Q_uP_v\right)\right)\cos\left((j-1)\lambda_u-(j-1)\lambda_v\right)\right.$$

$$\left.-\left(X_u\left(Q_uP_v-P_uQ_v\right)+Y_u\left(P_uP_v+Q_uQ_v\right)\right)\sin\left((j-1)\lambda_u-(j-1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}H_v^4}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{4}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j+1,u,v)}$$

$$\times\left(\left(X_u\left(P_uP_v+Q_uQ_v\right)+Y_u\left(P_uQ_v-Q_uP_v\right)\right)\cos\left((j+1)\lambda_u-(j+1)\lambda_v\right)\right.$$

$$\left.+\left(X_u\left(Q_uP_v-P_uQ_v\right)-Y_u\left(P_uP_v+Q_uQ_v\right)\right)\sin\left((j+1)\lambda_u-(j+1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^4}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{4}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j-1,u,v)}$$

$$\times\,((X_u\,(P_uP_v+Q_uQ_v)+Y_u\,(Q_uP_v-P_uQ_v))\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$-(X_u\,(Q_uP_v-P_uQ_v)-Y_u\,(P_uP_v+Q_uQ_v))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j-1,u,v)}$$

$$\times\,((X_v\,(P_uP_v+Q_uQ_v)+Y_v\,(P_uQ_v-Q_uP_v))\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$-(X_v\,(Q_uP_v-P_uQ_v)+Y_v\,(P_uP_v+Q_uQ_v))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j+1,u,v)}$$

$$\times\,((X_v\,(P_uP_v+Q_uQ_v)+Y_v\,(P_uQ_v-Q_uP_v))\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$+(X_v\,(Q_uP_v-P_uQ_v)+Y_v\,(P_uP_v+Q_uQ_v))\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_u^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j+1,u,v)}$$

$$\times\,((X_v\,(P_uP_v+Q_uQ_v)-Y_v\,(P_uQ_v-Q_uP_v))\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$-(X_v\,(P_uQ_v-Q_uP_v)+Y_v\,(P_uP_v+Q_uQ_v))\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j-1,u,v)}$$

$$\times\,((X_v\,(P_uP_v+Q_uQ_v)-Y_v\,(P_uQ_v-Q_uP_v))\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$+(X_v\,(P_uQ_v-Q_uP_v)+Y_v\,(P_uP_v+Q_uQ_v))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{H_v^5}\frac{1}{8}\frac{\beta_v^2}{\beta_u^2}H_u^2\, b_{3/2}^{(j,u,v)}\,((P_v^2-Q_v^2)\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$-2P_vQ_v\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{H_v^5}\frac{1}{8}\frac{\beta_v^2}{\beta_u^2}H_u^2\, b_{3/2}^{(j,u,v)}\,((P_v^2-Q_v^2)\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$+2P_vQ_v\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}j-\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times((X_u(P_v^2-Q_v^2)-2Y_uP_vQ_v)\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$-(2X_uP_vQ_v+Y_u(P_v^2-Q_v^2))\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}-\frac{1}{8}j-\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times((X_u(P_v^2-Q_v^2)-2Y_uP_vQ_v)\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$+(2X_uP_vQ_v+Y_u(P_v^2-Q_v^2))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}j-\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times((X_u(P_v^2-Q_v^2)+2Y_uP_vQ_v)\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$+(2X_uP_vQ_v-Y_u(P_v^2-Q_v^2))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}-\frac{1}{8}\,j-\frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,\big((X_u\,(P_v^2-Q_v^2)+2Y_u\,P_v\,Q_v)\cos\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)$$

$$+\,(-2X_u\,P_v\,Q_v+Y_u\,(P_v^2-Q_v^2))\sin\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)\big)$$

$$+\frac{1}{H^5\,\sqrt{H_v}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}\,j+\frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,\big((X_v\,(P_v^2-Q_v^2)-2Y_v\,P_v\,Q_v)\cos\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)$$

$$-\,(2X_v\,P_v\,Q_v+Y_v\,(P_v^2-Q_v^2))\sin\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)\big)$$

$$+\frac{1}{H_v^5\,\sqrt{H_v}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}\,j+\frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,\big((X_v\,(P_v^2-Q_v^2)-2Y_v\,P_v\,Q_v)\cos\left((j-1)\,\lambda_u-(j-1)\,\lambda_v\right)$$

$$+\,(2X_v\,P_v\,Q_v+Y_v\,(P_v^2-Q_v^2))\sin\left((j-1)\,\lambda_u-(j-1)\,\lambda_v\right)\big)$$

$$+\frac{1}{H_v^5\,\sqrt{H_v}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}\,j+\frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,\big((X_v\,(P_v^2-Q_v^2)+2Y_v\,P_v\,Q_v)\cos\left((j-1)\,\lambda_u-(j-1)\,\lambda_v\right)$$

$$+\,(2X_v\,P_v\,Q_v-Y_v\,(P_v^2-Q_v^2))\sin\left((j-1)\,\lambda_u-(j-1)\,\lambda_v\right)\big)$$

$$+\frac{1}{H_v^5\,\sqrt{H_v}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}\,j+\frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,\big((X_v\,(P_v^2-Q_v^2)+2Y_v\,P_v\,Q_v)\cos\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)$$

$$+\,(-2X_v\,P_v\,Q_v+Y_v\,(P_v^2-Q_v^2))\sin\left((j+1)\,\lambda_u-(j+1)\,\lambda_v\right)\big)$$

$$+\frac{1}{H_u\, H_v^4}\,\frac{1}{8}\,\frac{\beta_v^2}{\beta_u^2}\,H_u^2\, b_{3/2}^{(j,u,v)}\,\left((P_u^2 - Q_u^2)\cos\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right.$$

$$\left. - 2P_u Q_u \sin\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\, H_v^4}\,\frac{1}{8}\,\frac{\beta_v^2}{\beta_u^2}\,H_u^2\, b_{3/2}^{(j,u,v)}\,\left((P_u^2 - Q_u^2)\cos\left((j+1)\lambda_u - (j+1)\lambda_v\right)\right.$$

$$\left. + 2P_u Q_u \sin\left((j+1)\lambda_u - (j+1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_u}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8} + \frac{1}{8}\,j - \frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right) H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left((X_u(P_u^2 - Q_u^2) - 2Y_u P_u Q_u)\cos\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right.$$

$$\left. - (2X_u P_u Q_u + Y_u(P_u^2 - Q_u^2))\sin\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_u}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8} - \frac{1}{8}\,j - \frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right) H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left((X_u(P_u^2 - Q_u^2) - 2Y_u P_u Q_u)\cos\left((j+1)\lambda_u - (j+1)\lambda_v\right)\right.$$

$$\left. + (2X_u P_u Q_u + Y_u(P_u^2 - Q_u^2))\sin\left((j+1)\lambda_u - (j+1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16} - \frac{1}{8}\,j + \frac{1}{32}\,H_u\,\frac{\partial}{\partial H_u}\right) H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left((X_v(P_u^2 - Q_u^2) - 2Y_v P_u Q_u)\cos\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right.$$

$$\left. - (2X_v P_u Q_u + Y_v(P_u^2 - Q_u^2))\sin\left((j-1)\lambda_u - (j-1)\lambda_v\right)\right)$$

$$+\frac{1}{H_u\sqrt{H_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}j+\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\ ((X_v(P_u^2-Q_u^2)-2Y_vP_uQ_u)\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$+(2X_vP_uQ_u+Y_v(P_u^2-Q_u^2))\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}j+\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\ ((X_v(P_u^2-Q_u^2)+2Y_vP_uQ_u)\cos((j+1)\lambda_u-(j+1)\lambda_v)$$

$$+(2X_vP_uQ_u-Y_v(P_u^2-Q_u^2))\sin((j+1)\lambda_u-(j+1)\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}j+\frac{1}{32}H_u\frac{\partial}{\partial H_u}\right)H_u^2\, b_{3/2}^{(j,u,v)}$$

$$\times\ ((X_v(P_u^2-Q_u^2)+2Y_vP_uQ_u)\cos((j-1)\lambda_u-(j-1)\lambda_v)$$

$$-(2X_vP_uQ_u-Y_v(P_u^2-Q_u^2))\sin((j-1)\lambda_u-(j-1)\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\sqrt{H_v}\,H_v^4}\,\frac{-1}{4}\,\frac{\beta_v^2}{\beta_u^2}\,H_u^2\, b_{3/2}^{(j,u,v)}\,((P_uP_v-Q_uQ_v)\cos(j\lambda_u-j\lambda_v)$$

$$-(P_uQ_v+Q_uP_v)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^4}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{4}-\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,((X_u\,(P_u\,P_v-Q_u\,Q_v)-Y_u\,(P_u\,Q_v+Q_u\,P_v))\cos\,(j\,\lambda_u-j\,\lambda_v)$$

$$-\,(X_u\,(P_u\,Q_v+Q_u\,P_v)+Y_u\,(P_u\,P_v-Q_u\,Q_v))\sin\,(j\,\lambda_u-j\,\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^4}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}+\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,((X_u\,(P_u\,P_v-Q_u\,Q_v)-Y_u\,(P_u\,Q_v+Q_u\,P_v))\cos\,(j\,\lambda_u-j\,\lambda_v)$$

$$+\,(X_u\,(P_u\,Q_v+Q_u\,P_v)+Y_u\,(P_u\,P_v-Q_u\,Q_v))\sin\,(j\,\lambda_u-j\,\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^4}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{4}-\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,((X_u\,(P_u\,P_v-Q_u\,Q_v)+Y_u\,(P_u\,Q_v+Q_u\,P_v))\cos\,(j\,\lambda_u-j\,\lambda_v)$$

$$+\,(X_u\,(P_u\,Q_v+Q_u\,P_v)-Y_u\,(P_u\,P_v-Q_u\,Q_v))\sin\,(j\,\lambda_u-j\,\lambda_v))$$

$$+\frac{1}{H_u\sqrt{H_v}\,H_v^4}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}+\frac{1}{4}j+\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,((X_u\,(P_u\,P_v-Q_u\,Q_v)+Y_u\,(P_u\,Q_v+Q_u\,P_v))\cos\,(j\,\lambda_u-j\,\lambda_v)$$

$$-\,(X_u\,(P_u\,Q_v+Q_u\,P_v)-Y_u\,(P_u\,P_v-Q_u\,Q_v))\sin\,(j\,\lambda_u-j\,\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-3}{8}+\frac{1}{8}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\,((X_v\,(P_u\,P_v-Q_u\,Q_v)-Y_v\,(P_u\,Q_v+Q_u\,P_v))\cos\,(j\,\lambda_u-j\,\lambda_v)$$

$$-\,(X_v\,(P_u\,Q_v+Q_u\,P_v)+Y_v\,(P_u\,P_v-Q_u\,Q_v))\sin\,(j\,\lambda_u-j\,\lambda_v))$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\left((X_v(P_uP_v-Q_uQ_v)-Y_v(P_uQ_v+Q_uP_v))\cos(j\lambda_u-j\lambda_v)\right.$$

$$\left.+(X_v(P_uQ_v+Q_uP_v)+Y_v(P_uP_v-Q_uQ_v))\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-3}{8}+\frac{1}{8}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\left((X_v(P_uP_v-Q_uQ_v)+Y_v(P_uQ_v+Q_uP_v))\cos(j\lambda_u-j\lambda_v)\right.$$

$$\left.+(X_v(P_uQ_v+Q_uP_v)-Y_v(P_uP_v-Q_uQ_v))\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{1}{\sqrt{H_u}\,H_v^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}j-\frac{1}{16}H_u\frac{\partial}{\partial H_u}\right)H_u^2\,b_{3/2}^{(j,u,v)}$$

$$\times\left((X_v(P_uP_v-Q_uQ_v)+Y_v(P_uQ_v+Q_uP_v))\cos(j\lambda_u-j\lambda_v)\right.$$

$$\left.-(X_v(P_uQ_v+Q_uP_v)-Y_v(P_uP_v-Q_uQ_v))\sin(j\lambda_u-j\lambda_v)\right) \qquad (23).$$

5. LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATION OF THE DETERMINING FUNCTION S_{1p} WHICH ELIMINATES THE SHORT PERIOD TERMS

Its expression obtained through the classical way of Von Zeipel's method is:

$$\frac{\partial F_0}{\partial H'_u}\frac{\partial S_{1p}}{\partial \lambda_u}+\frac{\partial F_0}{\partial H'_v}\frac{\partial S_{1p}}{\partial \lambda_v}+\frac{\partial F_0}{\partial X'_u}\frac{\partial S_{1p}}{\partial Y_u}+\frac{\partial F_0}{\partial X'_v}\frac{\partial S_{1p}}{\partial Y_v}+\frac{\partial F_0}{\partial P'_u}\frac{\partial S_{1p}}{\partial Q_u}+\frac{\partial F_0}{\partial P'_v}\frac{\partial S_{1p}}{\partial Q_v}$$

$$-\frac{\partial F_0}{\partial Y_u}\frac{\partial S_{1p}}{\partial X'_u}-\frac{\partial F_0}{\partial Y_v}\frac{\partial S_{1p}}{\partial X'_v}-\frac{\partial F_0}{\partial Q_u}\frac{\partial S_{1p}}{\partial P'_u}-\frac{\partial F_0}{\partial Q_v}\frac{\partial S_{1p}}{\partial P'_v}=-F_1^* \qquad (24)$$

with

$$F_0=\frac{k^4\,m_0^2}{2}\sum_{u=1}^{n}\frac{\beta_u^3}{L'^2_u} \qquad (25)$$

and F^*_{1p} = set of the short period terms of F_{1p} obtained from (23), H'_u, H'_v, X'_u, X'_v, P'_u, P'_v being the new Hori variables which correspond to the old ones H_u, H_v, X_u, X_v, P_u, P_v in the canonical change of variables defined by the equality

$$F(H_u, H_v, X_u, X_v, P_u, P_v, \lambda_u, \lambda_v, Y_u, Y_v, Q_u, Q_v) = F'(H'_u, H'_v, X'_u, X'_v, P'_u, P'_v, Y'_u, Y'_v, Q'_u, Q'_v)$$

F' being the new hamiltonian which no more depends upon the λ's. From (25) we obtain:

$$\frac{\partial F_0}{\partial L'_u} = \frac{-k^4 m_0^2 \beta_u^3}{L'^3_u} .$$

From (25) and from the equality (6) written with the new variables L'_u, H'_u, X'_u, P'_u, Y'_u, Q'_u we obtain:

$$\frac{\partial F_0}{\partial H'_u} = \frac{\partial F_0}{\partial L'_u}, \quad \frac{\partial F_0}{\partial X'_u} = \frac{\partial F_0}{\partial L'_u} X'_u, \quad \frac{\partial F_0}{\partial P'_u} = \frac{\partial F_0}{\partial L'_u} P'_u, \quad \frac{\partial F_0}{\partial Y_u} = \left(\frac{\partial F_0}{\partial Y'_u}\right)_{Y'_u = Y_u} = \frac{\partial F_0}{\partial L'_u} Y_u,$$

$$\frac{\partial F_0}{\partial Q_u} = \left(\frac{\partial F_0}{\partial Q'_u}\right)_{Q'_u = Q_u} = \frac{\partial F_0}{\partial L'_u} Q_u$$

and the similar equalities in the index ν. (24) may therefore be written:

$$\frac{-k^4 m_0^2 \beta_u^3}{L'^3_u} \left(\frac{\partial S_{1p}}{\partial \lambda_u} + X'_u \frac{\partial S_{1p}}{\partial Y_u} + P'_u \frac{\partial S_{1p}}{\partial Q_u} - Y_u \frac{\partial S_{1p}}{\partial X'_u} - Q_u \frac{\partial S_{1p}}{\partial P'_u}\right)$$

$$\frac{-k^4 m_0^2 \beta_v^3}{L'^3_v} \left(\frac{\partial S_{1p}}{\partial \lambda_v} + X'_v \frac{\partial S_{1p}}{\partial Y_v} + P'_v \frac{\partial S_{1p}}{\partial Q_v} - Y_v \frac{\partial S_{1p}}{\partial X'_v} - Q_v \frac{\partial S_{1p}}{\partial P'_v}\right) = -F^*_{1p} \qquad (26).$$

F^*_{1p} being a sum of expressions of the form $A\cos(q\lambda_u - q\lambda_v) + B\sin(q\lambda_u - q\lambda_v)$ with $q \neq 0$ and A, B independent of λ_u and λ_v, we are led, according to (26), to solve the linear first order partial differential equation

$$\frac{-k^4 m_0^2 \beta_u^3}{L'^3_u} \left(\frac{\partial S_{1p}}{\partial \lambda_u} + X'_u \frac{\partial S_{1p}}{\partial Y_u} + P'_u \frac{\partial S_{1p}}{\partial Q_u} - Y_u \frac{\partial S_{1p}}{\partial X'_u} - Q_u \frac{\partial S_{1p}}{\partial P'_u}\right)$$

$$\frac{-k^4 m_0^2 \beta_v^3}{L'^3_v} \left(\frac{\partial S_{1p}}{\partial \lambda_v} + X'_v \frac{\partial S_{1p}}{\partial Y_v} + P'_v \frac{\partial S_{1p}}{\partial Q_v} - Y_v \frac{\partial S_{1p}}{\partial X'_v} - Q_v \frac{\partial S_{1p}}{\partial P'_v}\right) = A\cos(q\lambda_u - q\lambda_v)$$

$$+ B\sin(q\lambda_u - q\lambda_v) \qquad (27)$$

A and B being functions of X'_u, X'_v, Y_u, Y_v, P'_u, P'_v, Q_u, Q_v, H'_u, H'_v.

6. THE INTEGRALS OF THE SYSTEM OF CHARACTERISTICS OF THE LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATION OF S_{1p}

The coefficients of the linear first order partial differential equation (27) are nomore constants as in the case of the Delaunay variables; they are functions of the Hori variables themselves but this situation may be easily overcome. The system of equations of the characteristics of (27) is:

$$\frac{d\lambda_u}{-\dfrac{k^4 m_0^2 \beta_u^3}{L_u'^3}} = \frac{dY_u}{-\dfrac{k^4 m_0^2 \beta_u^3}{L_u'^3} X_u'} = \frac{dQ_u}{-\dfrac{k^4 m_0^2 \beta_u^3}{L_u'^3} P_u'} = \frac{dX_u'}{\dfrac{k^4 m_0^2 \beta_u^3}{L_u'^3} Y_u} = \frac{dP_u'}{\dfrac{k^4 m_0^2 \beta_u^3}{L_u'^3} Q_u}$$

$$= \frac{d\lambda_v}{-\dfrac{k^4 m_0^2 \beta_v^3}{L_v'^3}} = \frac{dY_v}{-\dfrac{k^4 m_0^2 \beta_v^3}{L_v'^3} X_v'} = \frac{dQ_v}{-\dfrac{k^4 m_0^2 \beta_v^3}{L_v'^3} P_v'} = \frac{dX_v'}{\dfrac{k^4 m_0^2 \beta_v^3}{L_v'^3} Y_v} = \frac{dP_v'}{\dfrac{k^4 m_0^2 \beta_v^3}{L_v'^3} Q_v}$$

$$= \frac{dS_{1p}}{A \cos (q\lambda_u - q\lambda_v) + B \sin (q\lambda_u - q\lambda_v)} \qquad (28).$$

We obtain at once, from (28), the four integrals:

$$Y_u^2 + X_u'^2 = k_1, \quad Q_u^2 + P_u'^2 = k_2, \quad Y_v^2 + X_v'^2 = h_1, \quad Q_v^2 + P_v'^2 = h_2 \qquad (29)$$

k_1, k_2, h_1, h_2 being arbitrary constants.

On the other hand, from (6) written with the new Hori variables L_u', H_u', X_u', P_u' and the old ones Y_u, Q_u we obtain:

$$L_u'^{-3} \sim H_u'^{-3} - \frac{3}{2} H_u'^{-4} (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) + \frac{3}{2} H_u'^{-5} (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2)^2 + \cdots \qquad (30)$$

and the similar equality with the index v instead of u.

We have therefore, with respect to (27) in which L_u', L_v', H_u', H_v' are considered as constants and according to (29), the two supplementary integrals:

$$L_u'^{-3} \sim H_u'^{-3} - \frac{3}{2} H_u'^{-4} (k_1 + k_2) + \frac{3}{2} H_u'^{-5} (k_1 + k_2)^2 + \cdots = M \qquad (31),$$

$$L_v'^{-3} \sim H_v'^{-3} - \frac{3}{2} H_v'^{-4} (k_1 + k_2) + \frac{3}{2} H_v'^{-5} (k_1 + k_2)^2 + \cdots = N \qquad (32)$$

M, N being arbitrary constants.

From (28), (31), (32) we obtain a third supplementary integral:

$$\lambda_v - \frac{\beta_v^3 L_v'^{-3}}{\beta_u^3 L_u'^{-3}} \lambda_u = j_1 \qquad (33)$$

j_1 being an arbitrary constant.

Moreover, from (28) and (29) we obtain the four other supplementary integrals:

$$\arc\sin \frac{Y_u}{\sqrt{Y_u^2 + X_u'^2}} - \lambda_u = K_1, \quad \arc\sin \frac{Q_u}{\sqrt{Q_u^2 + P_u'^2}} - \lambda_u = K_2,$$

$$\arc\sin \frac{Y_v}{\sqrt{Y_v^2 + X_v'^2}} - \lambda_v = H_1, \quad \arc\sin \frac{Q_v}{\sqrt{Q_v^2 + P_v'^2}} - \lambda_v = H_2 \tag{34}$$

K_1, K_2, H_1, H_2 being arbitrary constants. (28) admits therefore the four integrals (29), the two integrals (31) and (32), the integral (33) and the four integrals (34) that is to say eleven integrals.

From (29) and (33) we obtain:

$$X_u' = \sqrt{k_1}\cos(\lambda_u + K_1), \quad P_u' = \sqrt{k_2}\cos(\lambda_u + K_2), \quad Y_u = \sqrt{k_1}\sin(\lambda_u + K_1), \quad Q_u = \sqrt{k_2}\sin(\lambda_u + K_2),$$

$$X_v' = \sqrt{h_1}\cos(\lambda_u + H_1), \quad P_v' = \sqrt{h_2}\cos(\lambda_v + H_2), \quad Y_v = \sqrt{h_1}\sin(\lambda_v + H_1), \quad Q_v = \sqrt{h_2}\sin(\lambda_v + H_2). \tag{35}$$

7. DETERMINATION OF S_{1p}

Each equation (27) is characterized by the pair of coefficients A, B. There are 65 such pairs, that is to say 65 equations (27). We solve each of those 65 equations. S_{1p} is the sum of the 65 solutions we thus obtain.

1. We shall develop in a detailed manner the calculation for the pair of coefficients $A = A_0\, X_u'$, $B = \pm A_0\, Y_u$, A_0 being a function of H_u', H_v', $X_u'^2 + Y_u^2$, $X_v'^2 + Y_v^2$, $P_u'^2 + Q_u^2$, $P_v'^2 + Q_v^2$. We have, according to (28), (31), (32), (33):

$$\frac{dS_{1p}}{d\lambda_u} = \frac{1}{-k^4 m_0^2 \beta_u^3 M}\left(A_0 X_u' \cos\left(q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)\lambda_u - qj_1\right) \pm A_0 Y_u \sin\left(q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)\lambda_u - qj_1\right)\right)$$

that is to say, in virtue of the first and third equalities (35):

$$\frac{dS_{1p}}{d\lambda_u} = \frac{A_0\sqrt{k_1}}{-k^4 m_0^2 \beta_u^3 M}\cos\left(\left(1 \mp q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)\right)\lambda_u + K_1 \pm qj_i\right)$$

whence:

$$S_{1p} = \frac{A_0\sqrt{k_1}}{-k^4 m_0^2 \beta_u^3 M}\; \frac{\sin\left(\left(1 \mp q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)\right)\lambda_u + K_1 \pm qj_1\right)}{1 \mp q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)} + Z_1 \tag{36}$$

Z_1 being an arbitrary constant.

We have, according to (33):

$$\left(1 \mp q\left(1 - \frac{\beta_v^3 N}{\beta_u^3 M}\right)\right) \lambda_u + k_1 \pm q j_1 = \mp q\lambda_u \pm q\lambda_v + \lambda_u + K_1$$

whence, according to the first and third equalities (35):

$$\sin(\mp q\lambda_u \pm q\lambda_v + \lambda_u + K_1) = \frac{X'_u}{\sqrt{k_1}} \sin(\mp q\lambda_u \mp q\lambda_v) + \frac{Y_u}{\sqrt{k_1}} \cos(\mp q\lambda_u \pm q\lambda_v) \qquad (37).$$

The general solution S_{1p} is therefore in this case, according to (29), (31), (32), (33), (34):

$$S_{1p} = \frac{A_0}{-k^4 m_0^2} \frac{X'_u \sin(\mp q\lambda_u \pm q\lambda_v) + Y_u \cos(\mp q\lambda_u \pm q\lambda_v)}{(1 \mp q)\beta_u^3 L_u'^{-3} \pm q\beta_v^3 L_v'^{-3}}$$
$$+ F\Bigg(X_u'^2 + Y_u^2,\ X_v'^2 + Y_v^2,\ P_u'^2 + Q_u^2,\ P_v'^2 + Q_v^2,\ L_u'^{-3},\ L_v'^{-3},$$
$$\lambda_v - \frac{\beta_v^3 L_v'^{-3}}{\beta_u^3 L_u'^{-3}} \lambda_u,\ \arcsin\frac{Y_u}{\sqrt{X_u'^2 + Y_u^2}} - \lambda_u,\ \arcsin\frac{Y_v}{\sqrt{X_v'^2 + Y_v^2}} - \lambda_v,$$
$$\arcsin\frac{Q_u}{\sqrt{P_u'^2 + Q_u^2}} - \lambda_u,\ \arcsin\frac{Q_v}{\sqrt{P_v'^2 + Q_v^2}} - \lambda_v\Bigg) \qquad (38)$$

F being an arbitrary function of its arguments. We shall assume $F = 0$ and we shall consider separately the two cases $A = A_0 X'_u$, $B = A_0 Y_u$ and $A = A_0 X'_u$, $B = -A_0 Y_u$. According to (30) and the similar equality with the index v instead of u, S_{1p} is, in the case 1, the index q being replaced by the index j:

$$S_{1p} = \frac{A_0}{-k^4 m_0^2} \frac{Y_u \cos(j\lambda_u - j\lambda_v) - X'_u \sin(j\lambda_u - j\lambda_v)}{(1-j)\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}$$
$$\times \left(1 + \frac{\frac{3}{2}(1-j)\beta_u^3 H_u'^{-4}(X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) + \frac{3}{2} j \beta_v^3 H_v'^{-4}(X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2)}{(1-j)\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}\right)$$

and, in the case 2:

$$S'_{1p} = \frac{A_0}{-k^4 m_0^2} \frac{Y_u \cos(j\lambda_u - j\lambda_v) + X'_u \sin(j\lambda_u - j\lambda_v)}{(1+j)\beta_u^3 H_u'^{-3} - j\beta_v^3 H_v'^{-3}}$$
$$\times \left(1 + \frac{\frac{3}{2}(1+j)\beta_u^3 H_u'^{-4}(X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) - \frac{3}{2} j \beta_v^3 H_v'^{-4}(X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2)}{(1+j)\beta_u^3 H_u'^{-3} - j\beta_v^3 H_v'^{-3}}\right)$$

A_0 being a function of H'_u, H'_v, $X'^2_u + Y^2_u$, $X'^2_v + Y^2_v$, $P'^2_u + Q^2_u$, $P'^2_v + Q^2_v$ in the part

$$\frac{A_0}{-k^4 m_0^2} \frac{Y_u \cos(j\lambda_u - j\lambda_v) \mp X'_u \sin(j\lambda_u - j\lambda_v)}{(1 \mp j)\beta_u^3 H'^{-3}_u \pm j\beta_v^3 H'^{-3}_v}$$

of S_{1p} and a function of H'_u, H'_v in the part

$$\frac{A_0}{-k^4 m_0^2} \frac{Y_u \cos(j\lambda_u - j\lambda_v) \mp X'_u \sin(j\lambda_u - j\lambda_v)}{((1 \mp j)\beta_u^3 H'^{-3}_u \pm j\beta_v^3 H'^{-3}_v)^2}$$

$$\left(\frac{3}{2}(1+j)\beta_u^3 H'^{-4}_u (X'^2_u + Y^2_u + P'^2_u + Q^2_u) - \frac{3}{2} j \beta_v^3 H'^{-4}_v (X'^2_v + Y^2_v + P'^2_v + Q^2_v)\right) \text{ of } S_{1p}.$$

2. A similar procedure is applied for each of the other 64 pairs of coefficients A, B. Summing up the 65 S_{1p} thus obtained, we have finally:

$$S_{1p} = \frac{-\sigma}{m_0} \sum_{\substack{u \neq v \\ 1 \leq u < v \leq n}} \beta_u \beta_v^3 \sum_{j=0}^{p} \Bigg[\Bigg\{ \left(\frac{1}{H'^2_v} - \frac{1}{H'^3_v}(X'^2_v + Y^2_v + P'^2_v + Q^2_v)\right) b_{1/2}^{(j,u,v)}$$

$$+ \left(\frac{X'^2_u + Y^2_u}{H'_u H'^2_v} + \frac{X'^2_v + Y^2_v}{H'^3_v}\right)\left(-j^2 + \frac{3}{16} H'_u \frac{\partial}{\partial H'_u} + \frac{1}{16} H'^2_u \frac{\partial^2}{\partial H'^2_u}\right) b_{1/2}^{(j,u,v)}$$

$$+ \left(\frac{-P'^2_u - Q^2_u}{8 H'_u H'^4_v} + \frac{-P'^2_v - Q^2_v}{8 H'^5_v}\right) \frac{\beta_v^2}{\beta_u^2} H'^2_u \left(b_{3/2}^{(j-1,u,v)} + b_{3/2}^{(j+1,u,v)}\right)$$

$$+ \frac{-\frac{3}{2} j \beta_u^3 H'^{-4}_u (X'^2_u + Y^2_u + P'^2_u + Q^2_u) + \frac{3}{2} j \beta_v^3 H'^{-4}_v (X'^2_v + Y^2_v + P'^2_v + Q^2_v)}{H'^2_v (-j\beta_u^3 H'^{-3}_u + j\beta_v^3 H'^{-3}_v)} b_{1/2}^{(j,u,v)} \Bigg\}$$

$$\times \frac{\sin(j\lambda_u - j\lambda_v)}{-j\beta_u^3 H'^{-3}_u + j\beta_v^3 H'^{-3}_v}$$

$$+ \Bigg\{ \frac{1}{\sqrt{H'_u}\, H'^2_v} \left(j - \frac{j}{H'_v}(X'^2_v + Y^2_v + P'^2_v + Q^2_v)\right.$$

$$\left. + \left(\frac{-1}{8}(2H'_u + X'^2_u + Y^2_u + P'^2_u + Q^2_u) + \frac{H'_u}{4H'_v}(X'^2_v + Y^2_v + P'^2_v + Q^2_v)\right) \frac{\partial}{\partial H'_u}\right) b_{1/2}^{(j,u,v)}$$

$$+ \frac{X'^2_u + Y^2_u}{H'_u \sqrt{H'_u}\, H'^2_v} \left(\frac{-1}{8} j - \frac{5}{8} j^2 - \frac{1}{2} j^3 + \left(\frac{13}{128} + \frac{3}{16} j + \frac{1}{8} j^2\right) H'_u \frac{\partial}{\partial H'_u}\right.$$

$$\left. + \left(\frac{-1}{128} + \frac{1}{32} j\right) H'^2_u \frac{\partial^2}{\partial H'^2_u} - \frac{1}{128} H'^3_u \frac{\partial^3}{\partial H'^3_u}\right) b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v'^2+Y_v^2}{H_u'\sqrt{H_u'}\,H_v'^2}\left(-j^3+\left(\frac{-3}{64}+\frac{3}{16}j+\frac{1}{4}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{-5}{64}+\frac{1}{16}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{64}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\left(\frac{P_u'^2+Q_u^2}{4H_u'\sqrt{H_u'}\,H_v'^4}+\frac{P_v'^2+Q_v^2}{4H_v'^5\sqrt{H_v'}}\right)\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{2}j+\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}\right)H_u'^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)$$

$$+\frac{1}{\sqrt{H_u'}\,H_v'^2}\left(j-\frac{1}{4}H_u'\frac{\partial}{\partial H_u'}\right)b_{3/2}^{(j,u,v)}$$

$$\left.\frac{\frac{3}{2}(1-j)\,\beta_u^3H_u'^{-4}\left(X_u'^2+Y_u^2+P_u'^2+Q_u^2\right)+\frac{3}{2}j\,\beta_v^3H_v'^{-4}\left(X_v'^2+Y_v^2+P_v'^2+Q_v^2\right)}{(1-j)\,\beta_u^3H_u'^{-3}+j\,\beta_v^3H_v'^{-3}}\right\}$$

$$\times\frac{Y_u\cos(j\lambda_u-j\lambda_v)-X_u'\sin(j\lambda_u-j\lambda_v)}{(1-j)\,\beta_u^3H_u'^{-3}+j\,\beta_v^3H_v'^{-3}}$$

$$+\left\{\frac{1}{\sqrt{H_u'}\,H_v'^2}\left(-j+\frac{j}{H_v'}\left(X_v'^2+Y_v^2+P_v'^2+Q_v^2\right)\right.\right.$$

$$\left.+\left(\frac{-1}{8}\left(2H_u'+X_u'^2+Y_u^2+P_u'^2+Q_u^2\right)+\frac{H_u'}{4H_v'}\left(X_v'^2+Y_v^2+P_v'^2+Q_v^2\right)\right)\frac{\partial}{\partial H_u'}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_u'^2+Y_u^2}{H_u'\sqrt{H_u'}\,H_v'^2}\left(\frac{1}{8}j-\frac{5}{8}j^2+\frac{1}{2}j^3+\left(\frac{13}{128}-\frac{3}{16}j+\frac{1}{8}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{1}{32}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{128}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X_v'^2+Y_v^2}{H_u'\sqrt{H_u'}\,H_v'^2}\left(j^3+\left(\frac{-3}{64}-\frac{3}{16}j+\frac{1}{4}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{-5}{64}-\frac{1}{16}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{64}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$+\left(\frac{P_u'^2+Q_u^2}{4H_u'\sqrt{H_u'}\,H_v'^4}+\frac{P_v'^2+Q_v^2}{4H_v'^5\sqrt{H_v'}}\right)\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{2}j+\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}\right)H_u'^2\left(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}\right)$$

$$+\frac{1}{\sqrt{H'_u}\,H'^2_v}\left(-j-\frac{1}{4}H'_u\frac{\partial}{\partial H'_u}\right)b_{3/2}^{(j,u,v)}$$

$$\times\frac{\frac{3}{2}(1+j)\,\beta_u^3 H'^{-4}_u\,(X'^2_u+Y^2_u+P'^2_u+Q^2_u)-\frac{3}{2}j\,\beta_v^3 H'^{-4}_v\,(X'^2_v+Y^2_v+P'^2_v+Q^2_v)}{(1+j)\,\beta_u^3 H'^{-3}_u-j\,\beta_v^3 H'^{-3}_v}\Bigg\}$$

$$\times\frac{Y_u\cos(j\lambda_u-j\lambda_v)+X'_u\sin(j\lambda_u-j\lambda_v)}{(1+j)\,\beta_u^3 H'^{-3}_u-j\,\beta_v^3 H'^{-3}_v}$$

$$+\Bigg\{\frac{1}{\sqrt{H'_v}\,H'^2_v}\left(\frac{1}{2}-j-\frac{1}{H'_v}\left(\frac{1}{2}-j\right)\right.(X'^2_v+Y^2_v+P'^2_v+Q^2_v)$$

$$+\left.\left(\frac{1}{8}(2H'_u+X'^2_u+Y^2_u+P'^2_u+Q^2_u)-\frac{H'_u}{4H'_v}(X'^2_v+Y^2_v+P'^2_v+Q^2_v)\right)\frac{\partial}{\partial H'_u}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X'^2_u+Y^2_u}{H'_u\sqrt{H'_v}\,H'^2_v}\left(-\frac{1}{2}j^2+j^3+\left(\frac{9}{64}-\frac{3}{16}j-\frac{1}{4}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$+\left.\left(\frac{7}{64}-\frac{1}{16}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{64}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b_{1/2}^{(j,u,v)}$$

$$+\frac{X'^2_v+Y^2_v}{H'^3_v\sqrt{H'_v}}\left(-\frac{1}{16}+\frac{5}{16}j-\frac{7}{8}j^2+\frac{1}{2}j^3+\left(\frac{17}{128}-\frac{1}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$+\left.\left(\frac{11}{128}-\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b_{1/2}^{(j,u,v)}$$

$$+\left(\frac{P'^2_u+Q^2_u}{4H'_u\sqrt{H'_v}\,H'^4_v}+\frac{P'^2_v+Q^2_v}{4H'^5_v\sqrt{H'_v}}\right)\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{4}+\frac{1}{2}j-\frac{1}{8}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,(b_{3/2}^{(j-1,u,v)}+b_{3/2}^{(j+1,u,v)}$$

$$+\frac{1}{\sqrt{H'_v}\,H'^2_v}\left(\frac{1}{2}-j+\frac{1}{4}H'_u\frac{\partial}{\partial H'_u}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{-\frac{3}{2}j\,\beta_u^3 H'^{-4}_u\,(X'^2_u+Y^2_u+P'^2_u+Q^2_u)+\frac{3}{2}(1+j)\,\beta_v^3 H'^{-4}_v\,(X'^2_v+Y^2_v+P'^2_v+Q^2_v)}{-j\,\beta_u^3 H'^{-3}_u+(1+j)\,\beta_v^3 H'^{-3}_v}\Bigg\}$$

$$\times\frac{Y_v\cos(j\lambda_u-j\lambda_v)-X'_v\sin(j\lambda_u-j\lambda_v)}{-j\,\beta_u^3 H'^{-3}_u+(1+j)\,\beta_v^3 H'^{-3}_v}$$

$$+ \left\{ \frac{1}{\sqrt{H'_v}\, H'^2_v} \left(\frac{1}{2} + j - \frac{1}{H'_v} \left(\frac{1}{2} + j \right) (X'^2_v + Y^2_v + P'^2_v + Q^2_v) \right. \right.$$

$$+ \left(\frac{1}{8} (2H'_u + X'^2_u + Y^2_u + P'^2_u + Q^2_u) - \frac{H'_u}{4H'_v} (X'^2_v + Y^2_v + P'^2_v + Q^2_v) \right) \left. \frac{\partial}{\partial H'_u} \right) b^{(j,u,v)}_{1/2}$$

$$+ \frac{X'^2_u + Y^2_u}{H'_u \sqrt{H'_v}\, H'^2_v} \left(-\frac{1}{2} j^2 - j^3 + \left(\frac{9}{64} + \frac{3}{16} j - \frac{1}{4} j^2 \right) H'_u \frac{\partial}{\partial H'_u} \right.$$

$$\left. + \left(\frac{7}{64} + \frac{1}{16} j \right) H'^2_u \frac{\partial^2}{\partial H'^2_u} + \frac{1}{64} H'^3_u \frac{\partial^3}{\partial H'^3_u} \right) b^{(j,u,v)}_{1/2}$$

$$+ \frac{X'^2_v + Y^2_v}{H'^3_v \sqrt{H'_v}} \left(\frac{-1}{16} - \frac{5}{16} j - \frac{7}{8} j^2 - \frac{1}{2} j^3 + \left(\frac{17}{128} - \frac{1}{8} j^2 \right) H'_u \frac{\partial}{\partial H'_u} \right.$$

$$\left. + \left(\frac{11}{128} + \frac{1}{32} j \right) H'^2_u \frac{\partial^2}{\partial H'^2_u} + \frac{1}{128} H'^3_u \frac{\partial^3}{\partial H'^3_u} \right) b^{(j,u,v)}_{1/2}$$

$$+ \left(\frac{P'^2_u + Q^2_u}{4H'_u \sqrt{H'_v}\, H'^4_v} + \frac{P'^2_v + Q^2_v}{4H'^5_v \sqrt{H'_v}} \right) \frac{\beta^2_v}{\beta^2_u} \left(\frac{-1}{4} - \frac{1}{2} j - \frac{1}{8} H'_u \frac{\partial}{\partial H'_u} \right) H'^2_u \left(b^{(j-1,u,v)}_{3/2} + b^{(j+1,u,v)}_{3/2} \right)$$

$$+ \frac{1}{\sqrt{H'_v}\, H'^2_v} \left(\frac{1}{2} + j + \frac{1}{4} H'_u \frac{\partial}{\partial H'_u} \right) b^{(j,u,v)}_{1/2}$$

$$\frac{\frac{3}{2} j \beta^3_u H'^{-4}_u (X'^2_u + Y^2_u + P'^2_u + Q^2_u) + \frac{3}{2} (1 - j) \beta^3_v H'^{-4}_v (X'^2_v + Y^2_v + P'^2_v + Q^2_v)}{j \beta^3_u H'^{-3}_u + (1 - j) \beta^3_v H'^{-3}_v}$$

$$\times \frac{Y_v \cos (j\lambda_u - j\lambda_v) + X'_v \sin (j\lambda_u - j\lambda_v)}{j \beta^3_u H'^{-3}_u + (1 - j) \beta^3_v H'^{-3}_v}$$

$$+ \frac{1}{H'_u H'^2_v} \left(-\frac{5}{8} j + \frac{1}{2} j^2 + \left(\frac{-5}{32} + \frac{1}{4} j \right) H'_u \frac{\partial}{\partial H'_u} + \frac{1}{32} H'^2_u \frac{\partial^2}{\partial H'^2_u} \right) b^{(j,u,v)}_{1/2}$$

$$\times \frac{2X'_u Y_u \cos (j\lambda_u - j\lambda_v) - (X'^2_u - Y^2_u) \sin (j\lambda_u - j\lambda_v)}{(2 - j) \beta^3_u H'^{-3}_u + j \beta^3_v H'^{-3}_v}$$

$$+\frac{1}{H_u' H_v'^2}\left(\frac{5}{8}j+\frac{1}{2}j^2+\left(\frac{-5}{32}-\frac{1}{4}j\right)H_u'\frac{\partial}{\partial H_u'}+\frac{1}{32}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{2X_u' Y_u \cos(j\lambda_u - j\lambda_v)+(X_u'^2-Y_u^2)\sin(j\lambda_u-j\lambda_v)}{(2+j)\beta_u^3 H_u'^{-3}-j\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H_u'}\sqrt{H_v'}H_v'^2}\left(-\frac{1}{2}j-j^2+\left(\frac{-3}{16}-\frac{1}{2}j\right)H_u'\frac{\partial}{\partial H_u'}-\frac{1}{16}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{(X_u' Y_v + Y_u X_v')\cos(j\lambda_u - j\lambda_v)-(X_u' X_v' - Y_u Y_v)\sin(j\lambda_u-j\lambda_v)}{(1-j)\beta_u^3 H_u'^{-3}+(1+j)\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H_u'}\sqrt{H_v'}H_v'^2}\left(\frac{1}{2}j-j^2+\left(\frac{-3}{16}+\frac{1}{2}j\right)H_u'\frac{\partial}{\partial H_u'}-\frac{1}{16}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{(X_u' Y_v + Y_u X_v')\cos(j\lambda_u - j\lambda_v)+(X_u' X_v' - Y_u Y_v)\sin(j\lambda_u-j\lambda_v)}{(1+j)\beta_u^3 H_u'^{-3}+(1-j)\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H_u'}\sqrt{H_v'}H_v'^2}\left(-\frac{1}{2}j+j^2-\frac{3}{16}H_u'\frac{\partial}{\partial H_u'}-\frac{1}{16}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{-(X_u' Y_v + Y_u X_v')\cos(j\lambda_u - j\lambda_v)-(X_u' X_v' + Y_u Y_v)\sin(j\lambda_u-j\lambda_v)}{(1-j)\beta_u^3 H_u'^{-3}-(1-j)\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H_u'}\sqrt{H_v'}H_v'^2}\left(\frac{1}{2}j+j^2-\frac{3}{16}H_u'\frac{\partial}{\partial H_u'}-\frac{1}{16}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{(-X_u' Y_v + Y_u X_v')\cos(j\lambda_u - j\lambda_v)+(X_u' X_v' + Y_u Y_v)\sin(j\lambda_u-j\lambda_v)}{(1+j)\beta_u^3 H_u'^{-3}-(1+j)\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{H_v'^3}\left(\frac{1}{2}+\frac{9}{8}j+\frac{1}{2}j^2+\left(\frac{11}{32}+\frac{1}{4}j\right)H_u'\frac{\partial}{\partial H_u'}+\frac{1}{32}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{2X_v' Y_v \cos(j\lambda_u - j\lambda_v)-(X_v'^2-Y_v^2)\sin(j\lambda_u-j\lambda_v)}{-j\beta_u^3 H_u'^{-3}+(2+j)\beta_v^3 H_v'^{-3}}$$

$$+\frac{1}{H_v'^3}\left(\frac{1}{2}-\frac{9}{8}j+\frac{1}{2}j^2+\left(\frac{11}{32}-\frac{1}{4}j\right)H_u'\frac{\partial}{\partial H_u'}+\frac{1}{32}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{2X_v'Y_v\cos(j\lambda_u-j\lambda_v)+(X_v'^2-Y_v^2)\sin(j\lambda_u-j\lambda_v)}{j\beta_u^3H_u'^{-3}+(2-j)\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H_u'\sqrt{H_u'}H_v'^2}\left(-\frac{13}{24}j+\frac{5}{8}j^2-\frac{1}{6}j^3+\left(\frac{-17}{128}+\frac{5}{16}j-\frac{1}{8}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{5}{128}-\frac{1}{32}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{384}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{Y_u(3X_u'^2-Y_u^2)\cos(j\lambda_u-j\lambda_v)-X_u'(X_u'^2-3Y_u^2)\sin(j\lambda_u-j\lambda_v)}{(3-j)\beta_u^3H_u'^{-3}+j\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H_u'\sqrt{H_u'}H_v'^2}\left(\frac{13}{24}j+\frac{5}{8}j^2+\frac{1}{6}j^3+\left(\frac{-17}{128}-\frac{5}{16}j-\frac{1}{8}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{5}{128}+\frac{1}{32}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{384}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{Y_u(3X_u'^2-Y_u^2)\cos(j\lambda_u-j\lambda_v)+X_u'(X_u'^2-3Y_u^2)\sin(j\lambda_u-j\lambda_v)}{(3+j)\beta_u^3H_u'^{-3}-j\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H_u'\sqrt{H_v'}H_v'^2}\left(-\frac{5}{16}j-\frac{3}{8}j^2+\frac{1}{2}j^3+\left(\frac{-15}{128}-\frac{1}{8}j+\frac{3}{8}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$\left.+\left(\frac{-1}{128}+\frac{3}{32}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}+\frac{1}{128}H_u'^3\frac{\partial^3}{\partial H_u'^3}\right)b_{1/2}^{(j,u,v)}$$

$$\times\frac{(2X_u'Y_uX_v'+(X_u'^2-Y_u^2)Y_v)\cos(j\lambda_u-j\lambda_v)-((X_u'^2-Y_u^2)X_v'-2X_u'Y_uY_v)\sin(j\lambda_u-j\lambda_v)}{(2-j)\beta_u^3H_u'^{-3}+(1+j)\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H'_u\sqrt{H'_v}\,H'^2_v}\left(\frac{5}{16}j-\frac{3}{8}j^2-\frac{1}{2}j^3+\left(\frac{-15}{128}+\frac{1}{8}j+\frac{3}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{3}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_uY_uX'_v+(X'^2_u-Y^2_u)Y_v)\cos(j\lambda_u-j\lambda_v)+((X'^2_u-Y^2_u)X'_v-2X'_uY_uY_v)\sin(j\lambda_u-j\lambda_v)}{(2+j)\beta^3_uH'^{-3}_u+(1-j)\beta^3_vH'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}\,H'^2_v}\left(-\frac{5}{16}j+\frac{7}{8}j^2-\frac{1}{2}j^3+\left(\frac{-15}{128}+\frac{3}{16}j-\frac{1}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-1}{128}+\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_uY_uX'_v-(X'^2_u-Y^2_u)Y_v)\cos(j\lambda_u-j\lambda_v)-((X'^2_u-Y^2_u)X'_v+2X'_uY_uY_v)\sin(j\lambda_u-j\lambda_v)}{(2-j)\beta^3_uH'^{-3}_u-(1-j)\beta^3_vH'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}\,H'^2_v}\left(\frac{5}{16}j+\frac{7}{8}j^2+\frac{1}{2}j^3+\left(\frac{-15}{128}-\frac{3}{16}j-\frac{1}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-1}{128}-\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_uY_uX'_v-(X'^2_u-Y^2_u)Y_v)\cos(j\lambda_u-j\lambda_v)+((X'^2_u-Y^2_u)X'_v+2X'_uY_uY_v)\sin(j\lambda_u-j\lambda_v)}{(2+j)\beta^3_uH'^{-3}_u-(1+j)\beta^3_vH'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^3_v}\left(-\frac{1}{2}j-\frac{9}{8}j^2-\frac{1}{2}j^3+\left(\frac{-27}{128}-\frac{11}{16}j-\frac{3}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-13}{128}-\frac{3}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}-\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_uX'_vY_v+Y_u(X'^2_v-Y^2_v))\cos(j\lambda_u-j\lambda_v)-(X'_u(X'^2_v-Y^2_v)-2Y_uX'_vY_v)\sin(j\lambda_u-j\lambda_v)}{(1-j)\beta^3_uH'^{-3}_u+(2+j)\beta^3_vH'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^3_v}\left(\frac{1}{2}j-\frac{9}{8}j^2+\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{11}{16}j-\frac{3}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-13}{128}+\frac{3}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}-\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_u X'_v Y_v+Y_u(X'^2_v-Y^2_v))\cos(j\lambda_u-j\lambda_v)+(X'_u(X'^2_v-Y^2_v)-2Y_u X'_v Y_v)\sin(j\lambda_u-j\lambda_v)}{(1+j)\beta^3_u H'^{-3}_u+(2-j)\beta^3_v H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^3_v}\left(\frac{1}{2}j+\frac{9}{8}j^2+\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{1}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-13}{128}-\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}-\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_u X'_v Y_v-Y_u(X'^2_v-Y^2_v))\cos(j\lambda_u-j\lambda_v)-(X'_u(X'^2_v-Y^2_v)+2Y_u X'_v Y_v)\sin(j\lambda_u-j\lambda_v)}{(-1-j)\beta^3_u H'^{-3}_u+(2+j)\beta^3_v H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^3_v}\left(-\frac{1}{2}j+\frac{9}{8}j^2-\frac{1}{2}j^3+\left(\frac{-27}{128}+\frac{1}{8}j^2\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{-13}{128}+\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}-\frac{1}{128}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{(2X'_u X'_v Y_v-Y_u(X'^2_v-Y^2_v))\cos(j\lambda_u-j\lambda_v)+(X'_u(X'^2_v-Y^2_v)+2Y_u X'_v Y_v)\sin(j\lambda_u-j\lambda_v)}{(-1+j)\beta^3_u H'^{-3}_u+(2-j)\beta^3_v H'^{-3}_v}$$

$$+\frac{1}{H'^3_v\sqrt{H'_v}}\left(\frac{27}{48}+\frac{65}{48}j+\frac{7}{8}j^2+\frac{1}{6}j^3+\left(\frac{59}{128}+\frac{1}{2}j\right)H'_u\frac{\partial}{\partial H'_u}\right.$$

$$\left.+\left(\frac{9}{128}+\frac{1}{32}j\right)H'^2_u\frac{\partial^2}{\partial H'^2_u}+\frac{1}{384}H'^3_u\frac{\partial^3}{\partial H'^3_u}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{Y_v(3X'^2_v-Y^2_v)\cos(j\lambda_u-j\lambda_v)-X'_v(X'^2_v-3Y^2_v)\sin(j\lambda_u-j\lambda_v)}{-j\,\beta^3_u H'^{-3}_u+(3+j)\beta^3_v H'^{-3}_v}$$

$$+\frac{1}{H'^{3}_{v}\sqrt{H'_{v}}}\left(\frac{27}{48}-\frac{65}{48}\,j+\frac{7}{8}\,j^{2}-\frac{1}{6}\,j^{3}+\left(\frac{59}{128}-\frac{1}{2}\,j\right)H'_{u}\frac{\partial}{\partial H'_{u}}\right.$$

$$\left.+\left(\frac{9}{128}-\frac{1}{32}\,j\right)H'^{2}_{u}\frac{\partial^{2}}{\partial H'^{2}_{u}}+\frac{1}{384}H'^{3}_{u}\frac{\partial^{3}}{\partial H'^{3}_{u}}\right)b^{(j,u,v)}_{1/2}$$

$$\times\frac{Y_{v}\,(3X'^{2}_{v}-Y^{2}_{v})\cos(j\lambda_{u}-j\lambda_{v})+X'_{v}\,(X'^{2}_{v}-3Y^{2}_{v})\sin(j\lambda_{u}-j\lambda_{v})}{j\,\beta^{3}_{u}H'^{-3}_{u}+(3-j)\,\beta^{3}_{v}H'^{-3}_{v}}$$

$$+\frac{1}{\sqrt{H'_{u}}\sqrt{H'_{v}}\,H'^{4}_{v}}\,\frac{1}{4}\,\frac{\beta^{2}_{v}}{\beta^{2}_{u}}\,H'^{2}_{u}\,b^{(j+1,u,v)}_{3/2}$$

$$\frac{(Q_{u}P'_{v}-P'_{u}Q_{v})\cos((j+1)\lambda_{u}-(j+1)\lambda_{v})-(P'_{u}P'_{v}+Q_{u}Q_{v})\sin((j+1)\lambda_{u}-(j+1)\lambda_{v})}{-j\,\beta^{3}_{u}H'^{-3}_{u}+j\,\beta^{3}_{v}H'^{-3}_{v}}$$

$$+\frac{1}{\sqrt{H'_{u}}\sqrt{H'_{v}}\,H'^{4}_{b}}\,\frac{1}{4}\,\frac{\beta^{2}_{v}}{\beta^{2}_{u}}\,H'^{2}_{u}\,b^{(j-1,u,v)}_{3/2}$$

$$\frac{(Q_{u}P'_{v}-P'_{u}Q_{v})\cos((j-1)\lambda_{u}-(j-1)\lambda_{v})+(P'_{u}P'_{v}+Q_{u}Q_{v})\sin((j-1)\lambda_{u}-(j-1)\lambda_{v})}{j\,\beta^{3}_{u}H'^{-3}_{u}-j\,\beta^{3}_{v}H'^{-3}_{v}}$$

$$+\frac{1}{H'_{u}\sqrt{H'_{v}}\,H'^{4}_{v}}\,\frac{\beta^{2}_{v}}{\beta^{2}_{u}}\left(-\frac{1}{4}\,j-\frac{1}{16}\,H'_{u}\frac{\partial}{\partial H'_{u}}\right)H'^{2}_{u}\,b^{(j+1,u,v)}_{3/2}$$

$$\times\frac{\begin{pmatrix}(X'_{u}\,(Q_{u}P'_{v}-P'_{u}Q_{v})+Y_{u}\,(P'_{u}P'_{v}+Q_{u}Q_{v}))\cos((j+1)\lambda_{u}-(j+1)\lambda_{v})\\-(X'_{u}\,(P'_{u}P'_{v}+Q_{u}Q_{v})+Y_{u}\,(P'_{u}Q_{v}-Q_{u}P'_{v}))\sin((j+1)\lambda_{u}-(j+1)\lambda_{v})\end{pmatrix}}{(1-j)\,\beta^{3}_{u}H'^{-3}_{u}+j\,\beta^{3}_{v}H'^{-3}_{v}}$$

$$+\frac{1}{H'_{u}\sqrt{H'_{v}}\,H'^{4}_{v}}\,\frac{\beta^{2}_{v}}{\beta^{2}_{u}}\left(\frac{1}{4}\,j-\frac{1}{16}\,H'_{u}\frac{\partial}{\partial H'_{u}}\right)H'^{2}_{u}\,b^{(j-1,u,v)}_{3/2}$$

$$\frac{\begin{pmatrix}(X'_{u}\,(Q_{u}P'_{v}-P'_{u}Q_{v})+Y_{u}\,(P'_{u}P'_{v}+Q_{u}Q_{v}))\cos((j-1)\lambda_{u}-(j-1)\lambda_{v})\\+(X'_{u}\,(P'_{u}P'_{v}+Q_{u}Q_{v})+Y_{u}\,(P'_{u}Q_{v}-Q_{u}P'_{v}))\sin((j-1)\lambda_{u}-(j-1)\lambda_{v})\end{pmatrix}}{(1+j)\,\beta^{3}_{u}H'^{-3}_{u}-j\,\beta^{3}_{v}H'^{-3}_{v}}$$

$$+\frac{1}{H'_u\sqrt{H'_v}\,H'^4_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j+1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(Q_u\,P'_v-P'_u\,Q_v)-Y_u\,(P'_u\,P'_v+Q_u\,Q_v))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\-(X'_u\,(P'_u\,P'_v+Q_u\,Q_v)-Y_u\,(P'_u\,Q_v-Q_u\,P'_v))\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{(-1-j)\,\beta_u^3\,H'^{-3}_u+j\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}\,H'^4_v}\,\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{4}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j-1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(Q_u\,P'_v-P'_u\,Q_v)-Y_u\,(P'_u\,P'_v+Q_u\,Q_v))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\+(X'_u\,(P'_u\,P'_v+Q_u\,Q_v)-Y_u\,(P'_u\,Q_v-Q_u\,P'_v))\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{(-1+j)\,\beta_u^3\,H'^{-3}_u-j\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^5_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j+1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_v\,(Q_u\,P'_v-P'_u\,Q_v)+Y_v\,(P'_u\,P'_v+Q_u\,Q_v))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\-(X'_v\,(P'_u\,P'_v+Q_u\,Q_v)+Y_v\,(P'_u\,Q_v-Q_u\,P'_v))\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{-j\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^5_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j-1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_v\,(Q_u\,P'_v-P'_u\,Q_v)+Y_v\,(P'_u\,P'_v+Q_u\,Q_v))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\+(X'_v\,(P'_u\,P'_v+Q_u\,Q_v)+Y_v\,(P'_u\,Q_v-Q_u\,P'_v))\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{j\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H_v'^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{4}j+\frac{1}{16}H'_u\frac{\partial}{\partial H'_u}\right)H_u'^2\,b_{3/2}^{(j-1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_v(P'_uQ_v-Q_uP'_v)+Y_v(P'_uP'_v+Q_uQ_v))\cos((j-1)\lambda_u-(j-1)\lambda_v)\\-(X'_v(P'_uP'_v+Q_uQ_v)-Y_v(P'_uQ_v-Q_uP'_v))\sin((j-1)\lambda_u-(j-1)\lambda_v)\end{pmatrix}}{-j\,\beta_u^3H_u'^{-3}+(1+j)\,\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H'_u}\,H_v'^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}j+\frac{1}{16}H'_u\frac{\partial}{\partial H'_u}\right)H_u'^2\,b_{3/2}^{(j+1,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_v(P'_uQ_v-Q_uP'_v)+Y_v(P'_uP'_v+Q_uQ_v))\cos((j+1)\lambda_u-(j+1)\lambda_v)\\+(X'_v(P'_uP'_v+Q_uQ_v)-Y_v(P'_uQ_v-Q_uP'_v))\sin((j+1)\lambda_u-(j+1)\lambda_v)\end{pmatrix}}{j\,\beta_u^3H_u'^{-3}+(1-j)\,\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H_v'^5}\frac{1}{8}\frac{\beta_v^2}{\beta_u^2}H_u'^2\,b_{3/2}^{(j,u,v)}\frac{2P'_vQ_v\cos((j-1)\lambda_u-(j-1)\lambda_v)-(P_v'^2-Q_v^2)\sin((j-1)\lambda_u-(j-1)\lambda_v)}{(1-j)\,\beta_u^3H_u'^{-3}+(1+j)\,\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{H_v'^5}\frac{1}{8}\frac{\beta_v^2}{\beta_u^2}H_u'^2\,b_{3/2}^{(j,u,v)}\frac{2P'_vQ_v\cos((j+1)\lambda_u-(j+1)\lambda_v)+(P_v'^2-Q_v^2)\sin((j+1)\lambda_u-(j+1)\lambda_v)}{(1+j)\,\beta_u^3H_u'^{-3}+(1-j)\,\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H'_u}\,H_v'^5}\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{8}-\frac{1}{8}j-\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H_u'^2\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_uP'_uQ_v+Y_u(P_v'^2-Q_v^2))\cos((j-1)\lambda_u-(j-1)\lambda_v)\\-(X'_u(P_v'^2-Q_v^2)-2Y_uP'_vQ_v)\sin((j-1)\lambda_u-(j-1)\lambda_v)\end{pmatrix}}{(2-j)\,\beta_u^3H_u'^{-3}+(1+j)\,\beta_v^3H_v'^{-3}}$$

$$+\frac{1}{\sqrt{H'_u}\,H_v'^5}\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}j-\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H_u'^2\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_uP'_vQ_v+Y_u(P_v'^2-Q_v^2))\cos((j+1)\lambda_u-(j+1)\lambda_v)\\+(X'_u(P_v'^2-Q_v^2)-2Y_uP'_vQ_v)\sin((j+1)\lambda_u-(j+1)\lambda_v)\end{pmatrix}}{(2+j)\,\beta_u^3H_u'^{-3}+(1-j)\,\beta_v^3H_v'^{-3}}$$

$$
+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}\,j-\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}
$$

$$
\times\frac{\begin{pmatrix}(2X'_u\,P'_v\,Q_v-Y_u\,(P'^{2}_v-Q_v^2))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\-(X'_u\,(P'^{2}_v-Q_v^2)+2Y_u\,P'_v\,Q_v)\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{-j\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}
$$

$$
+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{8}-\frac{1}{8}\,j-\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}
$$

$$
\times\frac{\begin{pmatrix}(2X'_u\,P'_v\,Q_v-Y_u\,(P'^{2}_v-Q_v^2))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\+(X'_u\,(P'^{2}_v-Q_v^2)+2Y_u\,P'_v\,Q_v)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{j\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}
$$

$$
+\frac{1}{\sqrt{H'_v}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}\,j+\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}
$$

$$
\times\frac{\begin{pmatrix}(2X'_v\,P'_v\,Q_v+Y_v\,(P'^{2}_v-Q_v^2))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\-(X'_v\,(P'^{2}_v-Q_v^2)-2Y_v\,P'_v\,Q_v)\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3\,H'^{-3}_u+(2+j)\,\beta_v^3\,H'^{-3}_v}
$$

$$
+\frac{1}{\sqrt{H'_v}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}\,j+\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}
$$

$$
\times\frac{\begin{pmatrix}(2X'_v\,P'_v\,Q_v+Y_v\,(P'^{2}_v-Q_v^2))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\+(X'_v\,(P'^{2}_v-Q_v^2)-2Y_v\,P'_v\,Q_v)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3\,H'^{-3}_u+(2-j)\,\beta_v^3\,H'^{-3}_v}
$$

$$
+\frac{1}{\sqrt{H'_v}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}\,j+\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}
$$

$$
\times\frac{\begin{pmatrix}(2X'_v\,P'_v\,Q_v-Y_v\,(P'^{2}_v-Q_v^2))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\-(X'_v\,(P'^{2}_v-Q_v^2)+2Y_v\,P'_v\,Q_v)\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3\,H'^{-3}_u+j\,\beta_v^3\,H'^{-3}_v}
$$

$$+\frac{1}{\sqrt{H'_v}\,H'^{5}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}\,j+\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)\,H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\,\frac{\begin{pmatrix}(2X'_v\,P'_v\,Q_v-Y_v\,(P'^{2}_v-Q_v^2))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\+(X'_v\,(P'^{2}_v-Q_v^2)+2Y_v\,P'_v\,Q_v)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3\,H'^{-3}_u-j\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,H'^{4}_v}\,\frac{1}{8}\,\frac{\beta_v^2}{\beta_u^2}\,b_{3/2}^{(j,u,v)}\,\frac{2P'_u\,Q_u\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)-(P'^{2}_u-Q_u^2)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)}{(1-j)\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,H'^{4}_v}\,\frac{1}{8}\,\frac{\beta_v^2}{\beta_u^2}\,b_{3/2}^{(j,u,v)}\,\frac{2P'_u\,Q_u\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)+(P'^{2}_u-Q_u^2)\sin((j-1)\,\lambda_u-(j}{(1+j)\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_u}}\,\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{8}-\frac{1}{8}\,j-\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)\,H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\,\frac{\begin{pmatrix}(2X'_u\,P'_u\,Q_u+Y_u\,(P'^{2}_u-Q_u^2))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\-(X'_u\,(P'^{2}_u-Q_u^2)-2Y_u\,P'_u\,Q_u)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{(4+j)\,\beta_u^3\,H'^{-3}_u+(-1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_u}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}\,j-\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)\,H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\,\frac{\begin{pmatrix}(2X'_u\,P'_u\,Q_u+Y_u\,(P'^{2}_u-Q_u^2))\cos((j-1)\,\lambda_u-(j-1)\,\lambda_v)\\+(X'_u\,(P'^{2}_u-Q_u^2)-2Y_u\,P'_u\,Q_u)\sin((j-1)\,\lambda_u-(j-1)\,\lambda_v)\end{pmatrix}}{(4-j)\,\beta_u^3\,H'^{-3}_u+(-1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_u}}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}+\frac{1}{8}\,j-\frac{1}{32}\,H'_u\,\frac{\partial}{\partial H'_u}\right)\,H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\,\frac{\begin{pmatrix}(2X'_u\,P'_u\,Q_u-Y_u\,(P'^{2}_u-Q_u^2))\cos((j+1)\,\lambda_u-(j+1)\,\lambda_v)\\-(X'_u\,(P'^{2}_u-Q_u^2)+2Y_u\,P'_u\,Q_u)\sin((j+1)\,\lambda_u-(j+1)\,\lambda_v)\end{pmatrix}}{(2+j)\,\beta_u^3\,H'^{-3}_u+(-1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_u}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}-\frac{1}{8}j-\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_u P'_u Q_u - Y_u(P'^2_u - Q_u^2))\cos((j-1)\lambda_u-(j-1)\lambda_v)\\ +(X'_u(P'^2_u-Q_u^2)+2Y_u P'_u Q_u)\sin((j-1)\lambda_u-(j-1)\lambda_v)\end{pmatrix}}{(2-j)\,\beta_u^3 H'^{-3}_u+(-1+j)\,\beta_v^3 H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}j+\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_v P'_u Q_u + Y_v(P'^2_u - Q_u^2))\cos((j+1)\lambda_u-(j+1)\lambda_v)\\ -(X'_v(P'^2_u-Q_u^2)-2Y_v P'_u Q_u)\sin((j+1)\lambda_u-(j+1)\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3 H'^{-3}_u+(2+j)\,\beta_v^3 H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}j+\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_v P'_u Q_u + Y_v(P'^2_u - Q_u^2))\cos((j-1)\lambda_u-(j-1)\lambda_v)\\ +(X'_v(P'^2_u-Q_u^2)-2Y_v P'_u Q_u)\sin((j-1)\lambda_u-(j-1)\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3 H'^{-3}_u+(2-j)\,\beta_v^3 H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{3}{16}-\frac{1}{8}j+\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_v P'_u Q_u - Y_v(P'^2_u - Q_u^2))\cos((j+1)\lambda_u-(j+1)\lambda_v)\\ -(X'_v(P'^2_u-Q_u^2)+2Y_v P'_u Q_u)\sin((j+1)\lambda_u-(j+1)\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3 H'^{-3}_u+j\,\beta_v^3 H'^{-3}_v}$$

$$+\frac{1}{H'_u\sqrt{H'_v}}\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{16}+\frac{1}{8}j+\frac{1}{32}H'_u\frac{\partial}{\partial H'_u}\right)H'^2_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(2X'_v P'_u Q_u - Y_v(P'^2_u - Q_u^2))\cos((j-1)\lambda_u-(j-1)\lambda_v)\\ +(X'_v(P'^2_u-Q_u^2)+2Y_v P'_u Q_u)\sin((j-1)\lambda_u-(j-1)\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3 H'^{-3}_u-j\,\beta_v^3 H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,\sqrt{H'_v}\,H'^{4}_v}\;\frac{-1}{4}\,\frac{\beta_v^2}{\beta_u^2}\,H'^{2}_u\,b_{3/2}^{(j,u,v)}\;\frac{(P'_u\,Q_v+Q_u\,P'_v)\cos(j\lambda_u-j\lambda_v)-(P'_u\,P'_v-Q_uQ_v)\sin(j\lambda_u-j\lambda_v)}{(1-j)\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,\sqrt{H'_v}\,H'^{4}_v}\;\frac{-1}{4}\,\frac{\beta_v^2}{\beta_u^2}\,H'^{2}_u\,b_{3/2}^{(j,u,v)}\;\frac{(P'_u\,Q_v+Q_u\,P'_v)\cos(j\lambda_u-j\lambda_v)+(P'_u\,P'_v-Q_uQ_v)\sin(j\lambda_u-j\lambda_v)}{(1+j)\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_v}\,H'^{4}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}+\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(P'_u\,Q_v+Q_u\,P'_v)+Y_u\,(P'_u\,P'_v-Q_u\,Q_v))\cos(j\lambda_u-j\lambda_v)\\-(X'_u\,(P'_u\,P'_v-Q_u\,Q_v)-Y_u\,(P'_u\,Q_v+Q_u\,P'_v))\sin(j\lambda_u-j\lambda_v)\end{pmatrix}}{(2-j)\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_v}\,H'^{4}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(-\frac{1}{4}-\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(P'_u\,Q_v+Q_u\,P'_v)+Y_u\,(P'_u\,P'_v-Q_u\,Q_v))\cos(j\lambda_u-j\lambda_v)\\+(X'_u\,(P'_u\,P'_v-Q_u\,Q_v)-Y_u\,(P'_u\,Q_v+Q_u\,P'_v))\sin(j\lambda_u-j\lambda_v)\end{pmatrix}}{(2+j)\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_v}\,H'^{4}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{4}-\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(P'_u\,Q_v+Q_u\,P'_v)-Y_u\,(P'_u\,P'_v-Q_u\,Q_v))\cos(j\lambda_u-j\lambda_v)\\-(X'_u\,(P'_u\,P'_v-Q_u\,Q_v)+Y_u\,(P'_u\,Q_v+Q_u\,P'_v))\sin(j\lambda_u-j\lambda_v)\end{pmatrix}}{-j\,\beta_u^3\,H'^{-3}_u+(1+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{H'_u\,\sqrt{H'_v}\,H'^{4}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{4}+\frac{1}{4}\,j+\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b_{3/2}^{(j,u,v)}$$

$$\times\frac{\begin{pmatrix}(X'_u\,(P'_u\,Q_v+Q_u\,P'_v)-Y_u\,(P'_u\,P'_v-Q_u\,Q_v))\cos(j\lambda_u-j\lambda_v)\\+(X'_u\,(P'_u\,P'_v-Q_u\,Q_v)+Y_u\,(P'_u\,Q_v+Q_u\,P'_v))\sin(j\lambda_u-j\lambda_v)\end{pmatrix}}{j\,\beta_u^3\,H'^{-3}_u+(1-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}$$

$$\times\frac{\begin{pmatrix}(X'_v\,(P'_u\,Q_v+Q_u\,P'_v)+Y_v\,(P'_u\,P'_v-Q_u\,Q_v))\cos\,(j\lambda_u-j\lambda_v)\\-\,(X'_v\,(P'_u\,P'_v-Q_u\,Q_v)-Y_v\,(P'_u\,Q_v+Q_u\,P'_v))\sin\,(j\lambda_u-j\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3\,H'^{-3}_u+(2+j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(-\frac{3}{8}+\frac{1}{8}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}$$

$$\times\frac{\begin{pmatrix}(X'_v\,(P'_u\,Q_v+Q_u\,P'_v)+Y_v\,(P'_u\,P'_v-Q_u\,Q_v))\cos\,(j\lambda_u-j\lambda_v)\\+\,(X'_v\,(P'_u\,P'_v-Q_u\,Q_v)-Y_v\,(P'_u\,Q_v+Q_u\,P'_v))\sin\,(j\lambda_u-j\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3\,H'^{-3}_u+(2-j)\,\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{-3}{8}+\frac{1}{8}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}$$

$$\times\frac{\begin{pmatrix}(X'_v\,(P'_u\,Q_v+Q_u\,P'_v)-Y_v\,(P'_u\,P'_v-Q_u\,Q_v))\cos\,(j\lambda_u-j\lambda_v)\\-\,(X'_v\,(P'_u\,P'_v-Q_u\,Q_v)+Y_v\,(P'_u\,Q_v+Q_u\,P'_v))\sin\,(j\lambda_u-j\lambda_v)\end{pmatrix}}{(1-j)\,\beta_u^3\,H'^{-3}_u+j\beta_v^3\,H'^{-3}_v}$$

$$+\frac{1}{\sqrt{H'_u}\,H'^{5}_v}\,\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}-\frac{1}{4}\,j-\frac{1}{16}\,H'_u\,\frac{\partial}{\partial H'_u}\right)H'^{2}_u\,b^{(j,u,v)}_{3/2}$$

$$\left.\times\frac{\begin{pmatrix}(X'_v\,(P'_u\,Q_v+Q_u\,P'_v)-Y_v\,(P'_u\,P'_v-Q_u\,Q_v))\cos\,(j\lambda_u-j\lambda_v)\\+\,(X'_v\,(P'_u\,P'_v-Q_u\,Q_v)+Y_v\,(P'_u\,Q_v+Q_u\,P'_v))\sin\,(j\lambda_u-j\lambda_v)\end{pmatrix}}{(1+j)\,\beta_u^3\,H'^{-3}_u-j\beta_v^3\,H'^{-3}_v}\right] \qquad (39).$$

8. FIRST ORDER PARTIAL DERIVATIVES OF S_{1p} WITH RESPECT TO THE HORI VARIABLES AND ELIMINATION OF THE FIRST SHORT PERIOD TERMS

The first order partial derivatives of S_{1p} with respect to X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v are obtained at once from (39) and we shall not write them. We shall no more write the first order partial derivatives of S_{1p} with respect to λ_u and λ_v which are also obtained at once from (39). All we have to do is to calculate the first order partial derivatives of S_{1p} with respect to H'_u and H'_v. Since terms of S_{1p} of order four in X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, $P'_v Q_v$ become terms of order three in the first order partial derivatives of S_{1p} with respect to X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v, we shall truncate the first order partial derivatives of S_{1p} with respect to H'_u and H'_v to the sum of their terms of order 0, 1, 2 in X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v. The first order partial derivatives of S_{1p} with respect to X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v and its first order partial derivatives with respect to λ_u, λ_v will therefore be also truncated to the sum of their terms of order 0, 1, 2 in X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v.

A glimpse at (39) shows us that the first order partial derivatives of S_{1p} with respect to H'_u and H'_v and of order 0, 1, 2 in X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v may be reduced to the calculation of the first order partial derivatives with respect to H'_u and H'_v of the three quantities:

$$U_1 = \frac{H'^{\alpha}_u H'^{\beta}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left(a + b H'_u \frac{\partial}{\partial H'_u} + c H'^2_u \frac{\partial^2}{\partial H'^2_u} \right) b^{(j,u,v)}_{1/2} \qquad (40),$$

$$U_2 = \frac{H'^{\alpha}_u H'^{\beta}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} a H'^2_u b^{(j+s,u,v)}_{3/2} \qquad (41),$$

$$U_3 = \frac{H'^{\alpha}_u H'^{\beta}_v}{(q_1 H'^{-3}_u + q_2 H'^{-3}_v)^2} (a H'^{-4}_u + b H'^{-4}_v) b^{(j,u,v)}_{1/2} \qquad (42)$$

with $s = 0, \pm 1$, the a's, b's, c's being independent of H'_u and H'_v, the α's, β's positive or negative rational integers and q_1, q_2 relative integers.

From (40), (41), (42) we obtain, after a short calculation:

$$\frac{\partial U_1}{\partial H'_u} = \frac{H'^{\alpha-1}_u H'^{\beta}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left[a\alpha + (a + (1+\alpha) b) H'_u \frac{\partial}{\partial H'_u} + (b + (2+\alpha) c) H'^2_u \frac{\partial^2}{\partial H'^2_u} + c H'^3_u \frac{\partial^3}{\partial H'^3_u} \right.$$
$$\left. + \frac{3q_1}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left(a H'^{-3}_u + b H'^{-2}_u \frac{\partial}{\partial H'_u} + c H'^{-1}_u \frac{\partial^2}{\partial H'^2_u} \right) \right] b^{(j,u,v)}_{1/2} \qquad (43),$$

$$\frac{\partial U_2}{\partial H'_u} = \frac{H'^{\alpha-1}_u H'^{\beta}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left[a\alpha + a H'_u \frac{\partial}{\partial H'_u} + \frac{3q_1}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} a H'^{-3}_u \right] H'^2_u b^{(j+s,u,v)}_{3/2} \qquad (44),$$

$$\frac{\partial U_3}{\partial H'_u} = \frac{H'^{\alpha-5}_u H'^{\beta-4}_v}{(q_1 H'^{-3}_u + q_2 H'^{-3}_v)^2} \left[a(\alpha-4) H'^4_v + b\alpha H'^4_u + (a H'_u H'^4_v + b H'^5_u) \frac{\partial}{\partial H'_u} \right.$$
$$\left. + \frac{6q_1}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} (a H'^4_v H'^{-3}_u + b H'_u) \right] b^{(j,u,v)}_{1/2} \qquad (45),$$

and:

$$\frac{\partial U_1}{\partial H'_v} = \frac{H'^{\alpha}_u H'^{\beta-1}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left[a\beta + a H'_v \frac{\partial}{\partial H'_v} + b\beta H'_u \frac{\partial}{\partial H'_u} + b H'_u H'_v \frac{\partial^2}{\partial H'_u \partial H'_v} + c\beta H'^2_u \frac{\partial^2}{\partial H'^2_u} \right.$$

$$+ c H'^2_u H'_v \frac{\partial^3}{\partial H'^2_u \partial H'_v}$$

$$\left. + \frac{3q_2}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left(a H'^{-3}_v + b H'^{-3}_v H'_u \frac{\partial}{\partial H'_u} + c H'^{-3}_u H'^2_u \frac{\partial^2}{\partial H'^2_u} \right) \right] b^{(j,u,v)}_{1/2} \quad (46),$$

$$\frac{\partial U_2}{\partial H'_v} = \frac{H'^{\alpha}_u H'^{\beta-1}_v}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} \left[a\beta + a H'_v \frac{\partial}{\partial H'_v} + \frac{3q_2}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} a H'^{-3}_v \right] H'^2_u b^{(j+s,u,v)}_{3/2} \quad (47),$$

$$\frac{\partial U_3}{\partial H'_v} = \frac{H'^{\alpha-4}_u H'^{\beta-5}_v}{(q_1 H'^{-3}_u + q_2 H'^{-3}_v)^2} \left[a\beta H'^4_v + b(\beta-4) H'^4_u + (a H'^5_v + b H'^4_u H'_v) \frac{\partial}{\partial H'_v} \right.$$

$$\left. + \frac{6q_2}{q_1 H'^{-3}_u + q_2 H'^{-3}_v} (a H'_v + b H'^4_u H'^{-3}_v) \right] b^{(j,u,v)}_{1/2} \quad (48).$$

We shall develop in a detailed manner the calculation of

$$\frac{\partial U_1}{\partial H'_u}, \quad \frac{\partial U_2}{\partial H'_u}, \quad \frac{\partial U_3}{\partial H'_u}$$

for the U_1, U_2, U_3 which appear in the coefficient of the term in $\sin(j\lambda_u - j\lambda_v)$, the calculation of $\partial U_1/\partial H'_u$ for the U_1 which appear in the coefficient of the term in $2X'_u Y_u \cos(j\lambda_u - j\lambda_v) - (X'^2_u - Y^2_u)\sin(j\lambda_u - j\lambda_v)$ and the calculation of $\partial U_2/\partial H'_u$ for the U_2 which appear in the coefficient of the term in $(Q_u P'_v - P'_u Q_v)\cos(j+1)\lambda_u - (j+1)\lambda_v) - (P'_u P'_v + Q_u Q_v)\sin((j+1)\lambda_u - (j+1)\lambda_v)$, the first of these three terms corresponding to the pair of coefficients $A_0, 0$; the second one corresponding to the pair of coefficients $A_0(X'^2_u - Y^2_u), -A_0 2X'_u Y_u$ and the third one corresponding to the pair of coefficients $A_0(P'_u P'_v + Q_u Q_v), -A_0(Q_u P'_v - P'_u Q_v)$.

The term in A_0, 0 contains four U_1, two U_2 and one U_3. The term in $A_0(X'^2_u - Y^2_u), -A_0 2X'_u Y_u$ contains one U_1, no U_2 and U_3. The term in $A_0(P'_u P'_v + Q_u Q_v), -A_0(Q_u P'_v - P'_u Q_v)$ contains one U_2, no U_1 and U_3.

One of the four U_1 of the term in $A_0, 0$ is:

$$U_1 = \frac{H'^{-1}_u H'^{-2}_v}{-j\beta^3_u H'^{-3}_u + j\beta^3_v H'^{-3}_v} (X'^2_u + Y^2_u) \left(-j^2 + \frac{3}{16} H'_u \frac{\partial}{\partial H'_u} + \frac{1}{16} H'^2_u \frac{\partial^2}{\partial H'^2_u} \right) b^{(j,u,v)}_{1/2} \quad (49).$$

One of its two U_2 is:

$$U_2 = \frac{H'^{-1}_u H'^{-4}_v}{-j\beta^3_u H'^{-3}_u + j\beta^3_v H'^{-3}_v} (-P'^2_u - Q^2_u) \frac{\beta^2_v}{\beta^2_u} \frac{1}{8} H'^2_u b^{(j-1,u,v)}_{3/2} \quad (50).$$

Its U_3 is:

$$U_3 = \frac{H_v'^{-2}}{(-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3})^2}\left(-\frac{3}{2} j\beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_u'^{-4} + \frac{3}{2} j\beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2) H_v'^{-4}\right) b_{1/2}^{(j,u,v)} \qquad (51).$$

(49), (50), (51) show that $q_1 = -j\beta_u^3$, $q_2 = j\beta_v^3$. In (49) we have: $\alpha = -1$, $\beta = -2$, $a = -(X_u'^2 + Y_u^2) j^2$, $b = (X_u'^2 + Y_u^2)$ 3/16, $c = (X_u'^2 + Y_u^2)$ 1/16. In (50) we have: $\alpha = -1$, $\beta = -4$, $s = -1$, $a = -(P_u'^2 + Q_u^2)$ 1/8 β_v^2/β_u^2. In (51) we have: $\alpha = 0$, $\beta = -2$, $\alpha = -3/2\ j\beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2)$, $b = 3/2\ j\beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2)$.

Whence, according to (43), (44), (45):

$$\frac{\partial U_1}{\partial H_u'} = \frac{H_u'^{-2} H_v'^{-2}}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} (X_u'^2 + Y_u^2)\left[j^2 - j^2 H_u' \frac{\partial}{\partial H_u'} + \frac{1}{4} H_u'^2 \frac{\partial^2}{\partial H_u'^2} + \frac{1}{16} H_u'^3 \frac{\partial^3}{\partial H_u'^3} + \frac{-3j\beta_u^3}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}\left(-j^2 H_u'^{-3} + \frac{3}{16} H_u'^{-2} \frac{\partial}{\partial H_u'} + \frac{1}{16} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2}\right)\right] b_{1/2}^{(j,u,v)},$$

$$\frac{\partial U_2}{\partial H_u'} = \frac{H_u'^{-2} H_v'^{-4}}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} (P_u'^2 + Q_u^2) \frac{\beta_v^2}{\beta_u^2}\left[\frac{1}{8} - H_u' \frac{\partial}{\partial H_u'} + \frac{-3j\beta_u^3}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \frac{-1}{8} H_u'^{-3}\right] \times H_u'^2\, b_{3/2}^{(j-1,u,v)},$$

$$\frac{\partial U_3}{\partial H_u'} = \frac{H_u'^{-5} H_v'^{-6}}{(-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3})^2}\Bigg[6j\beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_v'^4 + \left(\frac{-3}{2} j\beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_u' H_v'^4 + \frac{3}{2} j\beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2) H_u'^5\right)\frac{\partial}{\partial H_u'} + \frac{-6j\beta_u^3}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}\left(\frac{-3}{2} j\beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_v'^4 H_u'^{-3} + \frac{3}{2} j\beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2) H_u'\right)\Bigg] b_{1/2}^{(j,u,v)}.$$

The U_1 of the term in $A_0 (X_u'^2 - Y_u^2)$, $A_0 2X_u' Y_u$ is:

$$U_1 = \frac{H_u'^{-1} H_v'^{-2}}{(2-j)\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \left(-\frac{5}{8} j + \frac{1}{2} j^2 + \left(\frac{-5}{32} + \frac{1}{4} j\right) H_u' \frac{\partial}{\partial H_u'} + \frac{1}{32} H_u'^2 \frac{\partial^2}{\partial H_u'^2}\right) b_{1/2}^{(j,u,v)}$$

with

$$q_1 = (2-j)\beta_u^3, \quad q_2 = j\beta_v^3, \quad \alpha = -1, \quad \beta = -2, \; a = -\frac{5}{8} j + \frac{1}{2} j^2, \quad b = \frac{-5}{32} + \frac{1}{4} j, \quad c = \frac{1}{32}$$

Whence, according to (43):

$$\frac{\partial U_1}{\partial H_u'} = \frac{H_u'^{-2} H_v'^{-2}}{(2-j)\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \left[\frac{5}{8} j - \frac{1}{2} j^2 + \left(-\frac{5}{8} j + \frac{1}{2} j^2\right) H_u' \frac{\partial}{\partial H_u'} + \left(\frac{-1}{8} + \frac{1}{4} j\right) H_u'^2 \frac{\partial^2}{\partial H_u'^2}\right.$$

$$+ \frac{1}{32} H_u'^3 \frac{\partial^3}{\partial H_u'^3}$$

$$+ \frac{3(2-j)\beta_u^3}{(2-j)\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \left(\left(-\frac{5}{8} j + \frac{1}{2} j^2\right) H_u'^{-3} + \left(\frac{-5}{32} + \frac{1}{4} j\right) H_u'^{-2} \frac{\partial}{\partial H_u'}\right.$$

$$\left.\left. + \frac{1}{32} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2}\right)\right] b_{1/2}^{(j,u,v)}.$$

The U_2 of the term in $A_0 (P_u' P_v' + Q_u Q_v)$, $-A_0 (Q_u P_v' - P_u' Q_v)$ is:

$$U_2 = \frac{H_u'^{-1/2} H_v'^{-3/2}}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \frac{1}{4} \frac{\beta_v^2}{\beta_u^2} H_u'^2 b_{3/2}^{(j+1,u,v)}$$

with

$$q_1 = -j\beta_u^3, \quad q_2 = j\beta_v^3, \quad \alpha = -\frac{1}{2}, \quad \beta = -\frac{9}{2}, \quad s = 1, \quad a = \frac{1}{4} \frac{\beta_v^2}{\beta_u^2}$$

whence:

$$\frac{\partial U_2}{\partial H_u'} = \frac{H_u'^{-3/2} H_v'^{-9/2}}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} \frac{1}{4} \frac{\beta_v^2}{\beta_u^2} \left[-\frac{1}{2} + H_u' \frac{\partial}{\partial H_u'} + \frac{-3j\beta_u^3}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}} H_u'^{-3}\right] H_u'^2 b_{3/2}^{(j+1,u,v)}.$$

We should calculate in a similar manner the three others U_1 and the other U_2 of the term in $A_0, 0$ and the $\partial U_1/\partial H_u'$, $\partial U_2/\partial H_u'$, $\partial U_3/\partial H_u'$ of the other terms of S_{1p} which are of order 0, 1, 2 in X_u', Y_u, P_u', Q_u, X_v', Y_v, P_v', Q_v. Multiplying each of these partial derivatives by its corresponding factor in $A\cos(q\lambda_u - q\lambda_v) + B\sin(q\lambda_u - q\lambda_v)$ and summing up all the terms we thus obtain, we have finally:

$$\frac{\partial S_{1p}}{\partial H_u'} = \frac{-\sigma}{m_0} \sum_{\substack{u \neq v \\ 1 \le u < v \le n}} \beta_u \beta_v^3 \sum_{j=0}^{p} \left[\left[\frac{1}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}\right.\right.$$

$$\times \left\{(H_u'^{-1} H_v'^{-2} - H_u'^{-1} H_v'^{-3} (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2)) \left(H_u' \frac{\partial}{\partial H_u'} + \frac{-3j\beta_u^3 H_u'^{-3}}{-j\beta_u^3 H_u'^{-3} + j\beta_v^3 H_v'^{-3}}\right) b_{1/2}^{(j,u,v)}\right.$$

$$+ (X_u'^2 + Y_u^2) H_u'^{-2} H_v'^{-2} \left(j^2 - j^2 H_u' \frac{\partial}{\partial H_u'} + \frac{1}{4} H_u'^2 \frac{\partial^2}{\partial H_u'^2} + \frac{1}{16} H_u'^3 \frac{\partial^3}{\partial H_u'^3} \right.$$

$$\left. + \frac{-3 j \beta_u^3}{-j \beta_u^3 H_u'^{-3} + j \beta_v^3 H_v'^{-3}} \left(- j^2 H_u'^{-3} + \frac{3}{16} H_u'^{-2} \frac{\partial}{\partial H_u'} + \frac{1}{16} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2} \right) \right) b_{1/2}^{(j,u,v)}$$

$$+ (X_v'^2 + Y_v^2) H_u'^{-1} H_v'^{-3} \left(\left(- j^2 + \frac{3}{16} \right) H_u' \frac{\partial}{\partial H_u'} + \frac{5}{16} H_u'^2 \frac{\partial^2}{\partial H_u'^2} + \frac{1}{16} H_u'^3 \frac{\partial^3}{\partial H_u'^3} \right.$$

$$\left. + \frac{-3 j \beta_u^3}{-j \beta_u^3 H_u'^{-3} + j \beta_v^3 H_v'^{-3}} \left(- j^2 H_u'^{-3} + \frac{3}{16} H_u'^{-2} \frac{\partial}{\partial H_u'} + \frac{1}{16} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2} \right) \right) b_{1/2}^{(j,u,v)}$$

$$+ \frac{\beta_v^2}{\beta_u^2} \frac{-P_u'^2 - Q_u^2}{8} H_u'^{-2} H_v'^{-4} \left(-1 + H_u' \frac{\partial}{\partial H_u'} + \frac{-3 j \beta_u^3 H_u'^{-3}}{-3 j \beta_u^3 H_u'^{-3} + 3 j \beta_v^3 H_v'^{-3}} \right)$$

$$\times H_u'^2 \left(b_{3/2}^{(j-1,u,v)} + b_{3/2}^{(j+1,u,v)} \right)$$

$$+ \frac{\beta_v^2}{\beta_u^2} \frac{-P_v'^2 - Q_v^2}{8} H_u'^{-1} H_v'^{-5} \left(H_u' \frac{\partial}{\partial H_u'} + \frac{-3 j \beta_u^3 H_u'^{-3}}{-3 j \beta_u^3 H_u'^{-3} + 3 j \beta_v^3 H_v'^{-3}} \right)$$

$$\times H_u'^2 \left(b_{3/2}^{(j-1,u,v)} + b_{3/2}^{(j+1,u,v)} \right)$$

$$\frac{H_u'^{-5} H_v'^{-6}}{(-j \beta_u^3 H_u'^{-3} + j \beta_v^3 H_v'^{-3})^2} \left\{ 6 j \beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_v'^4 \right.$$

$$+ \left(-\frac{3}{2} j \beta_u^3 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) \; H_u' H_v'^4 \right.$$

$$\left. + \frac{3}{2} j \beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2) \; H_u'^5 \right) \frac{\partial}{\partial H_u'}$$

$$\left. \left. + \frac{\begin{pmatrix} 9 j^2 \beta_u^6 (X_u'^2 + Y_u^2 + P_u'^2 + Q_u^2) H_v'^4 H_u'^{-3} \\ - 9 j^2 \beta_u^3 \beta_v^3 (X_v'^2 + Y_v^2 + P_v'^2 + Q_v^2) H_u' \end{pmatrix}}{-j \beta_u^3 H_u'^{-3} + j \beta_v^3 H_v'^{-3}} \right\} \right] b_{1/2}^{(j,u,v)}$$

$$\times \sin (j \lambda_u - j \lambda_v)$$

$$
+\frac{H_u'^{-3/2}\,H_v'^{-2}}{(1-j)\,\beta_u^3\,H_u'^{-3}+j\beta_v^3\,H_v'^{-3}}\left(-\frac{1}{2}j+\left(\frac{-1}{8}+j\right)H_u'\frac{\partial}{\partial H_u'}-\frac{1}{4}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right.
$$

$$
\left.+\frac{3\,(1-j)\,\beta_u^3}{(1-j)\,\beta_u^3\,H_u'^{-3}+j\beta_v^3\,H_v'^{-3}}\left(jH_u'^{-3}-\frac{1}{4}H_u'^{-2}\frac{\partial}{\partial H_u'}\right)\right)b_{1/2}^{(j,u,v)}
$$

$$
\times\,(Y_u\cos(j\lambda_u-j\lambda_v)-X_u'\sin(j\lambda_u-j\lambda_v))
$$

$$
+\frac{H_u'^{-3/2}\,H_v'^{-2}}{(1+j)\,\beta_u^3\,H_u'^{-3}-j\beta_v^3\,H_v'^{-3}}\left(\frac{1}{2}j+\left(\frac{-1}{8}-j\right)H_u'\frac{\partial}{\partial H_u'}-\frac{1}{4}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right.
$$

$$
\left.+\frac{3\,(1+j)\,\beta_u^3}{(1+j)\,\beta_u^3\,H_u'^{-3}-j\beta_v^3\,H_v'^{-3}}\left(-jH_u'^{-3}-\frac{1}{4}H_u'^{-2}\frac{\partial}{\partial H_u'}\right)\right)b_{1/2}^{(j,u,v)}
$$

$$
\times\,(Y_u\cos(j\lambda_u-j\lambda_v)+X_u'\sin(j\lambda_u-j\lambda_v))
$$

$$
+\frac{H_u'^{-1}\,H_v'^{-5/2}}{-j\beta_u^3\,H_u'^{-3}+(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{3}{4}-j\right)H_u'\frac{\partial}{\partial H_u'}+\frac{1}{4}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right.
$$

$$
\left.+\frac{-3j\beta_u^3}{-j\beta_u^3\,H_u'^{-3}+(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{1}{2}-j\right)H_u'^{-3}+\frac{1}{4}H_u'^{-2}\frac{\partial}{\partial H_u'}\right)\right)b_{1/2}^{(j,u,v)}
$$

$$
\times\,(Y_v\cos(j\lambda_u-j\lambda_v)-X_v'\sin(j\lambda_u-j\lambda_v))
$$

$$
+\frac{H_u'^{-1}\,H_v'^{-5/2}}{j\beta_u^3\,H_u'^{-3}+(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{3}{4}+j\right)H_u'\frac{\partial}{\partial H_u'}+\frac{1}{4}H_u'^2\frac{\partial^2}{\partial H_u'^2}\right.
$$

$$
\left.+\frac{3j\beta_u^3}{j\beta_u^3\,H_u'^{-3}+(1-j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{1}{2}+j\right)H_u'^{-3}+\frac{1}{4}H_u'^{-2}\frac{\partial}{\partial H_u'}\right)\right)
$$

$$
\times\,(Y_v\cos(j\lambda_u-j\lambda_v)+X_v'\sin(j\lambda_u-j\lambda_v))
$$

$$+\frac{H_u'^{-2}\,H_v'^{-2}}{(2-j)\,\beta_u^3\,H_u'^{-3}+j\beta_v^3\,H_v'^{-3}}\left(\frac{5}{8}j-\frac{1}{2}j^2+\left(\frac{-5}{8}j+\frac{1}{2}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$+\left(\frac{-1}{8}+\frac{1}{4}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}+\frac{1}{32}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(2-j)\,\beta_u^3}{(2-j)\,\beta_u^3\,H_u'^{-3}+j\beta_v^3\,H_v'^{-3}}\left(\left(\frac{-5}{8}j+\frac{1}{2}j^2\right)H_u'^{-3}\right.$$

$$+\left(\frac{-5}{32}+\frac{1}{4}j\right)H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.+\frac{1}{32}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\,(2X_u'\,Y_u\cos(j\lambda_u-j\lambda_v)-(X_u'^2-Y_u^2)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{H_u'^{-2}\,H_v'^{-2}}{(2+j)\,\beta_u^3\,H_u'^{-3}-j\beta_v^3\,H_v'^{-3}}\left(\frac{-5}{8}j-\frac{1}{2}j^2+\left(\frac{5}{8}j+\frac{1}{2}j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$+\left(\frac{-1}{8}-\frac{1}{4}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}+\frac{1}{32}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(2+j)\,\beta_u^3}{(2+j)\,\beta_u^3\,H_u'^{-3}-j\beta_v^3\,H_v'^{-3}}\left(\left(\frac{5}{8}j+\frac{1}{2}j^2\right)H_u'^{-3}\right.$$

$$+\left(\frac{-5}{32}-\frac{1}{4}j\right)H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.+\frac{1}{32}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\,(2X_u'\,Y_u\cos(j\lambda_u-j\lambda_v)+(X_u'^2-Y_u^2)\sin(j\lambda_u-j\lambda_v))$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-5/2}}{(1-j)\,\beta_u^3\,H_u'^{-3}+(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\frac{1}{4}j+\frac{1}{2}j^2+\left(\frac{-3}{32}-\frac{3}{4}j-j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$+\left(\frac{-9}{32}-\frac{1}{2}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{16}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(1-j)\,\beta_u^3}{(1-j)\,\beta_u^3\,H_u'^{-3}+(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\left(-\frac{1}{2}j-j^2\right)H_u'^{-3}\right.$$

$$+\left(\frac{-3}{16}-\frac{1}{2}j\right)H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.-\frac{1}{16}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u'\,Y_v+Y_u\,X_v')\cos(j\lambda_u-j\lambda_v)-(X_u'\,X_v'-Y_u\,Y_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-5/2}}{(1+j)\,\beta_u^3\,H_u'^{-3}+(1-j)\,\beta_v^3\,H_v'^{-3}}\left(\frac{-1}{4}j+\frac{1}{2}j^2+\left(\frac{-3}{32}+\frac{3}{4}j-j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$+\left(\frac{-9}{32}+\frac{1}{2}j\right)H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{16}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(1+j)\,\beta_u^3}{(1+j)\,\beta_u^3\,H_u'^{-3}+(1-j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{1}{2}j-j^2\right)H_u'^{-3}\right.$$

$$+\left(\frac{-3}{16}+\frac{1}{2}j\right)H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.-\frac{1}{16}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((X_u'\,Y_v+Y_u\,X_v')\cos(j\lambda_u-j\lambda_v)+(X_u'\,X_v'-Y_u\,Y_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-5/2}}{(1-j)\,\beta_u^3\,H_u'^{-3}-(1-j)\,\beta_v^3\,H_v'^{-3}}\left(\frac{1}{4}j-\frac{1}{2}j^2+\left(\frac{-3}{32}-\frac{1}{2}j+j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$-\frac{9}{32}H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{16}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(1-j)\,\beta_u^3}{(1-j)\,\beta_u^3\,H_u'^{-3}-(1-j)\,\beta_v^3\,H_v'^{-3}}\left(\left(-\frac{1}{2}j+j^2\right)H_u'^{-3}\right.$$

$$-\frac{3}{16}H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.-\frac{1}{16}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((-X_u'Y_v+Y_uX_v')\cos(j\lambda_u-j\lambda_v)-(X_u'X_v'+Y_uY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-5/2}}{(1+j)\,\beta_u^3\,H_u'^{-3}-(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\frac{-1}{4}j-\frac{1}{2}j^2+\left(\frac{-3}{32}+\frac{1}{2}j+j^2\right)H_u'\frac{\partial}{\partial H_u'}\right.$$

$$-\frac{9}{32}H_u'^2\frac{\partial^2}{\partial H_u'^2}-\frac{1}{16}H_u'^3\frac{\partial^3}{\partial H_u'^3}$$

$$+\frac{3\,(1+j)\,\beta_u^3}{(1+j)\,\beta_u^3\,H_u'^{-3}-(1+j)\,\beta_v^3\,H_v'^{-3}}\left(\left(\frac{1}{2}j+j^2\right)H_u'^{-3}\right.$$

$$-\frac{3}{16}H_u'^{-2}\frac{\partial}{\partial H_u'}$$

$$\left.\left.-\frac{1}{16}H_u'^{-1}\frac{\partial^2}{\partial H_u'^2}\right)\right)b_{1/2}^{(j,u,v)}$$

$$\times\left((-X_u'Y_v+Y_uX_v')\cos(j\lambda_u-j\lambda_v)+(X_u'X_v'+Y_uY_v)\sin(j\lambda_u-j\lambda_v)\right)$$

$$
+ \frac{H_u'^{-1} H_v'^{-2}}{-j\beta_u^3 H_u'^{-3} + (2+j)\beta_v^3 H_v'^{-3}} \left(\left(\frac{27}{32} + \frac{11}{8} j + \frac{1}{2} j^2 \right) H_u' \frac{\partial}{\partial H_u'} \right.
$$

$$
+ \left(\frac{13}{32} + \frac{1}{4} j \right) H_u'^2 \frac{\partial^2}{\partial H_u'^2} + \frac{1}{32} H_u'^3 \frac{\partial^3}{\partial H_u'^3}
$$

$$
+ \frac{-3j\beta_u^3}{-j\beta_u^3 H_u'^{-3} + (2+j)\beta_v^3 H_v'^{-3}} \left(\left(\frac{1}{2} + \frac{9}{8} j + \frac{1}{2} j^2 \right) H_u'^{-3} \right.
$$

$$
+ \left(\frac{11}{32} + \frac{1}{4} j \right) H_u'^{-2} \frac{\partial}{\partial H_u'}
$$

$$
\left. \left. + \frac{1}{32} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2} \right) \right) b_{1/2}^{(j,u,v)}
$$

$$
\times \left(2X_v' Y_v \cos(j\lambda_u - j\lambda_v) - (X_v'^2 - Y_v^2) \sin(j\lambda_u - j\lambda_v) \right)
$$

$$
+ \frac{H_u'^{-1} H_v'^{-2}}{j\beta_u^3 H_u'^{-3} + (2-j)\beta_v^3 H_v'^{-3}} \left(\left(\frac{27}{32} - \frac{11}{8} j + \frac{1}{2} j^2 \right) H_u' \frac{\partial}{\partial H_u'} \right.
$$

$$
+ \left(\frac{13}{32} - \frac{1}{4} j \right) H_u'^2 \frac{\partial^2}{\partial H_u'^2} + \frac{1}{32} H_u'^3 \frac{\partial^3}{\partial H_u'^3}
$$

$$
+ \frac{3j\beta_u^3}{j\beta_u^3 H_u'^{-3} + (2-j)\beta_v^3 H_v'^{-3}} \left(\left(\frac{1}{2} - \frac{9}{8} j + \frac{1}{2} j^2 \right) H_u'^{-3} \right.
$$

$$
+ \left(\frac{11}{32} - \frac{1}{4} j \right) H_u'^{-2} \frac{\partial}{\partial H_u'}
$$

$$
\left. \left. + \frac{1}{32} H_u'^{-1} \frac{\partial^2}{\partial H_u'^2} \right) \right) b_{1/2}^{(j,u,v)}
$$

$$
\times \left(2X_u' Y_v \cos(j\lambda_u - j\lambda_v) + (X_v'^2 - Y_v^2) \sin(j\lambda_u - j\lambda_v) \right)
$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-9/2}}{-j\beta_u^3 H_u'^{-3}+j\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}+\frac{1}{4}H_u'\frac{\partial}{\partial H_u'}+\frac{-3j\beta_u^3}{-j\beta_u^3 H_u'^{-3}+j\beta_v^3 H_v'^{-3}}\;\frac{1}{4}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j+1,u,v)}$$

$$\times\left((Q_u P_v'-P_u' Q_v)\cos((j+1)\lambda_u-(j+1)\lambda_v)-(P_u' P_v'+Q_u Q_v)\sin((j+1)\lambda_u-(j+1)\lambda_v)\right)$$

$$+\frac{H_u'^{-3/2}\,H_v'^{-3/2}}{j\beta_u^3 H_u'^{-3}-j\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}+\frac{1}{4}H_u'\frac{\partial}{\partial H_u'}+\frac{3j\beta_u^3}{j\beta_u^3 H_u'^{-3}-j\beta_v^3 H_v'^{-3}}\;\frac{1}{4}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j-1,u,v)}$$

$$\times\left((Q_u P_v'-P_u' Q_v)\cos((j-1)\lambda_u-(j-1)\lambda_v)+(P_u' P_v'+Q_u Q_v)\sin((j-1)\lambda_u-(j-1)\lambda_v)\right)$$

$$+\frac{H_u'^{-1}\,H_v'^{-5}}{(1-j)\beta_u^3 H_u'^{-3}+(1+j)\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}+\frac{3(1-j)\beta_u^3}{(1-j)\beta_u^3 H_u'^{-3}+(1+j)\beta_v^3 H_v'^{-3}}\;\frac{1}{8}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left(2P_v' Q_v\cos((j-1)\lambda_u-(j-1)\lambda_v)-(P_v'^2-Q_v^2)\sin(j-1)\lambda_u-(j-1)\lambda_v)\right)$$

$$+\frac{H_u'^{-1}\,H_v'^{-5}}{(1+j)\beta_u^3 H_u'^{-3}+(1-j)\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}+\frac{3(1+j)\beta_u^3}{(1+j)\beta_u^3 H_u'^{-3}+(1-j)\beta_v^3 H_v'^{-3}}\;\frac{1}{8}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left(2P_v' Q_v\cos((j+1)\lambda_u-(j+1)\lambda_v)+(P_v'^2-Q_v^2)\sin((j+1)\lambda_u-(j+1)\lambda_v)\right)$$

$$+\frac{H_u'^{-2}\,H_v'^{-4}}{(1-j)\beta_u^3 H_u'^{-3}+(1+j)\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}+\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}+\frac{3(1-j)\beta_u^3}{(1-j)\beta_u^3 H_u'^{-3}+(1+j)\beta_v^3 H_v'^{-3}}\;\frac{1}{8}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left(2P_u' Q_u\cos((j+1)\lambda_u-(j+1)\lambda_v)-(P_u'^2-Q_u^2)\sin((j+1)\lambda_u-(j+1)\lambda_v)\right)$$

$$+\frac{H_u'^{-2}\,H_v'^{-4}}{(1+j)\beta_u^3 H_u'^{-3}+(1-j)\beta_v^3 H_v'^{-3}}\;\frac{\beta_v^2}{\beta_u^2}\left(\frac{-1}{8}+\frac{1}{8}H_u'\frac{\partial}{\partial H_u'}+\frac{3(1+j)\beta_u^3}{(1+j)\beta_u^3 H_u'^{-3}+(1-j)\beta_v^3 H_v'^{-3}}\;\frac{1}{8}H_u'^{-3}\right)H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\times\left(2P_u' Q_u\cos((j-1)\lambda_u-(j-1)\lambda_v)+(P_u'^2-Q_u^2)\sin((j-1)\lambda_u-(j-1)\lambda_v)\right)$$

$$+ \frac{H_u'^{-3/2}\, H_v'^{-5/2}}{(1-j)\,\beta_u^3\, H_u'^{-3} + (1+j)\,\beta_v^3\, H_v'^{-3}} \; \frac{\beta_v^2}{\beta_u^2} \left(\frac{1}{8} - \frac{1}{4} H_u' \frac{\partial}{\partial H_u'} \right.$$

$$\left. + \frac{3\,(1-j)\,\beta_u^3}{(1-j)\,\beta_u^3\, H_u'^{-3} + (1+j)\,\beta_v^3\, H_v'^{-3}} \; \frac{-1}{4} H_u'^{-3} \right) H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\times \left((P_u' Q_v + Q_u P_v') \cos (j\lambda_u - j\lambda_v) - (P_u' P_v' - Q_u Q_v) \sin (j\lambda_u - j\lambda_v) \right)$$

$$+ \frac{H_u'^{-3/2}\, H_v'^{-9/2}}{(1+j)\,\beta_u^3\, H_u'^{-3} + (1-j)\,\beta_v^3\, H_v'^{-3}} \; \frac{\beta_v^2}{\beta_u^2} \left(\frac{1}{8} - \frac{1}{4} H_u' \frac{\partial}{\partial H_u'} \right.$$

$$\left. + \frac{3\,(1+j)\,\beta_u^3}{(1+j)\,\beta_u^3\, H_u'^{-3} + (1-j)\,\beta_v^3\, H_v'^{-3}} \; \frac{-1}{4} H_u'^{-3} \right) H_u'^2\, b_{3/2}^{(j,u,v)}$$

$$\left. \times \left((P_u' Q_v + Q_u P_v') \cos (j\lambda_u - j\lambda_v) + (P_u' P_v' - Q_u Q_v) \sin (j\lambda_u - j\lambda_v) \right) \right] .$$

In order not to lengthen inordinately this paper, we shall not write the expression of the partial derivative of S_{1p} with respect to H_v', letting the reader to obtain it from the equalities (46), (47), (48) applied to the coefficient of each term $A \cos (q\lambda_u - q\lambda_v) + B \sin (q\lambda_u - q\lambda_v)$ of S_{1p} which is of order 0, 1, 2 with respect to X_u', Y_u, P_u', Q_u, X_v', Y_v, P_v', Q_v.

According to Von Zeipel's method, the old Hori canonical variables H_u, H_v, X_u, X_v, P_u, P_v, λ_u, λ_v, Y_u, Y_v, Q_u, Q_v are then connected to the new ones H_u', H_v', X_u', X_v', P_u', P_v', Y_u', Y_v', Q_u', Q_v' eliminating the short period terms which arise from the principal part F_{1p} of the disturbing function through the $(n-1)n/2$ sets of twelve equalities

$$H_u = H_u' + \frac{\partial S_{1p}}{\partial \lambda_u}, \qquad \lambda_u = \lambda_u' - \frac{\partial S_{1p}}{\partial H_u'},$$

$$X_u = X_u' + \frac{\partial S_{1p}}{\partial Y_u}, \qquad Y_u = Y_u' - \frac{\partial S_{1p}}{\partial X_u'},$$

$$P_u = P_u' + \frac{\partial S_{1p}}{\partial Q_u}, \qquad Q_u = Q_u' - \frac{\partial S_{1p}}{\partial P_u'};$$

$$H_v = H_v' + \frac{\partial S_{1p}}{\partial \lambda_v}, \qquad \lambda_v = \lambda_v' - \frac{\partial S_{1p}}{\partial H_v'},$$

$$X_v = X_v' + \frac{\partial S_{1p}}{\partial Y_v}, \qquad Y_v = Y_v' - \frac{\partial S_{1p}}{\partial X_v'},$$

$$P_v = P_v' + \frac{\partial S_{1p}}{\partial Q_v}, \qquad Q_v = Q_v' - \frac{\partial S_{1p}}{\partial P_v'}$$

with $1 \leq u < v \leq n$.

For each couple u, v of values of u and v, we have to solve, by the method of the "retour des suites" of Lagrange applied to functions of several variables, the above equations in λ_u and λ_v and to bring the values of λ_u and λ_v we thus obtain in the equations in H_u, X_u, P_u, Y_u, Q_u and in the equations in H_v, X_v, P_v, Y_v, Q_v.

9. CONCLUSION

1. This study of the elimination of the short period terms of a first order general planetary theory through Von Zeipel's method and the Hori canonical variables lead us, for the calculation of the determining function S_{1_p}, to a linear first order partial differential equation which has no more constant coefficients as it occurred with the Delaunay variables. The obtention of eleven integrals of its system of characteristics allowed us, as we showed, to overcome this difficulty. On the other hand, the introduction of the Hori canonical variables lead us, in a very natural way, to replace the classical Newcomb operator $D_{u,v} = \alpha_{u,v}\, d/d\alpha_{u,v}$ by the operator $H_w (\partial/\partial H_w)$ $(w = 1, 2, \ldots, n)$. The application of this new operator to particular cases of a first order general planetary theory requires a tabulation of the Laplace coefficients as functions of H'_u and H'_v instead of $\alpha_{u,v}$. Such a tabulation could easily be carried out through the equalities (8) and (9).

2. The Hori canonical variables X'_u, Y_u, X'_v, Y_v which are of the order of magnitude of the eccentricities according to the equalities (16) and the Hori canonical variables P'_v, Q_u, P'_v, Q_v which are of the order of magnitude of the sines of inclinations according to the equalities (17) appear in S_{1_p}, according to (39), in the form of algebraic mononomials and polynomials. The partial derivatives of S_{1_p} with respect to X'_u, Y_u, P'_u, Q_u, X'_v, Y_v, P'_v, Q_v do not contain therefore divisors in those variables and the complication due to the presence of small divisors in the partial derivatives of S_{1_p} with respect to the linear variables when we use Delaunay variables is thus avoided.

3. (39) shows also that each argument of the sines and cosines of the truncated Fourier series of the disturbing function F_{1_p}, of S_{1_p} and of the partial derivatives of S_{1_p} with respect to Hori variables is of the form $q\lambda_u - q\lambda_v$ which did not occur with the Poincaré variables.

10. REFERENCES

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